

On the Enumeration of Irreducible Polynomials over $\mathbb{GF}(q)$ with Prescribed Coefficients

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Abstract. For any prime power $q = p^r$, any positive integer $l < p$ and any integer $n \geq l$ coprime to p , we present an algorithm which outputs exact expressions in terms of the number of points over \mathbb{F}_{q^n} of certain affine varieties defined over \mathbb{F}_q , for the number of monic irreducible polynomials in $\mathbb{F}_q[x]$ of degree n for which the coefficients of x^{n-1}, \dots, x^{n-l} are prescribed. As well as computing examples of these varieties for $q = 5$, for $q = 2$ we give varieties (which are all curves) for $l \leq 7$ and compute the corresponding L-polynomials for $l = 4$ and $l = 5$, obtaining explicit formulae for these open problems for n odd, while for $q = 3$ we provide explicit formulae for $l = 3$. We also detail some of the computational challenges and theoretical questions arising from this approach, in the general case.

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1 Introduction

For $q = p^r$ a prime power let \mathbb{F}_q denote the finite field of q elements, and let $I_q(n)$ denote the number of monic irreducible polynomials in $\mathbb{F}_q[x]$ of degree n . A classical result due to Gauss [14, pp. 602-629] states that

$$I_q(n) = \frac{1}{n} \sum_{d|n} \mu(d) q^{n/d}.$$

A natural problem is to determine the number of monic irreducible polynomials in $\mathbb{F}_q[x]$ of degree n for which certain coefficients are prescribed. As Panario has stated [21, p. 115], “*The long-term goal here is to provide existence and counting results for irreducibles with any number of prescribed coefficients to any given values. This goal is completely out of reach at this time. Incremental steps seem doable, but it would be most interesting if new techniques were introduced to attack these problems.*”

An interesting subproblem of this long-term goal is to determine the number of monic irreducible polynomials in $\mathbb{F}_q[x]$ of degree n for which the first l coefficients have the prescribed values t_1, \dots, t_l , which we denote by $I_q(n, t_1, \dots, t_l)$. Although asymptotics for such subproblems have been obtained by Cohen [8], only very few exact results are known. In 1952 Carlitz gave formulae for $I_q(n, t_1)$ [5], while in 1990 Kuz'min gave formulae for $I_q(n, t_1, t_2)$ [23,24]; Cattell *et al.* later reproduced Kuz'min's results for the base field \mathbb{F}_2 , in 1999 [6]. In 2001 the three coefficient case $I_2(n, t_1, t_2, t_3)$ was solved, by Yucas and Mullen for n even [39] and by Fitzgerald and Yucas for n odd [12]. Formulae for $I_{2^r}(n, t_1, t_2)$ for all $r \geq 1$ were given in 2013 by Ri *et al.* [31]. Most recently, in 2016 Ahmadi *et al.* gave formulae for $I_{2^r}(n, 0, 0, 0)$ for all $r \geq 1$ [2].

Rather than study the above subproblem instances directly, the papers [6,39,12,31,2] all study a set of equivalent problems, namely counting the number of elements of \mathbb{F}_{q^n} with correspondingly prescribed traces. In particular, for $a \in \mathbb{F}_{q^n}$ the characteristic polynomial of a w.r.t. the extension $\mathbb{F}_{q^n}/\mathbb{F}_q$ is defined to be:

$$\prod_{i=0}^{n-1} (x - a^{q^i}) = x^n - T_1(a)x^{n-1} + T_2(a)x^{n-2} - \dots + (-1)^{n-1}T_{n-1}(a)x + (-1)^nT_n(a), \quad (1)$$

with $T_l : \mathbb{F}_{q^n} \rightarrow \mathbb{F}_q$, $1 \leq l \leq n$ the successive trace functions

$$\begin{aligned}
T_1(a) &= \sum_{i=0}^{n-1} a^{q^i}, \\
T_2(a) &= \sum_{0 \leq i_1 < i_2 \leq n-1} a^{q^{i_1} + q^{i_2}}, \\
T_3(a) &= \sum_{0 \leq i_1 < i_2 < i_3 \leq n-1} a^{q^{i_1} + q^{i_2} + q^{i_3}}, \\
&\vdots \\
T_l(a) &= \sum_{0 \leq i_1 < \dots < i_l \leq n-1} a^{q^{i_1} + \dots + q^{i_l}}, \\
&\vdots \\
T_n(a) &= a^{1+q+q^2+\dots+q^{n-1}}.
\end{aligned}$$

As is natural from the above definitions, we say that the degree of any trace function T_l is l . For any $l \leq n$ and $t_1, \dots, t_l \in \mathbb{F}_q$, let $F_q(n, t_1, \dots, t_l)$ be the number of elements $a \in \mathbb{F}_{q^n}$ for which $T_1(a) = t_1, \dots, T_l(a) = t_l$. If for a given $q = p^r$ and $l \leq n$ one determines all $F_q(n, t_1, \dots, t_l)$, then an elementary Möbius inversion-type argument gives all $I_q(n, t_1, \dots, t_l)$, and vice versa.

As well as proving formulae for $I_{2^r}(n, 0, 0, 0)$ for all $r \geq 1$, the work [2] also explained an intriguing phenomenon, which is that the formulae for $F_2(n, t_1, t_2)$ and $F_2(n, t_1, t_2, t_3)$ proven in [6] and [39,12], depend on $n \bmod 8$ and on $n \bmod 24$, respectively. In particular, by Fourier analysing the formulae, they were shown to be related to the number of \mathbb{F}_{2^n} -rational points of certain genus one and two supersingular curves defined over \mathbb{F}_2 . A simple argument gave a new derivation of the formulae for the $t_1 = 0$ cases and the odd n cases, and since the curves featured are supersingular their normalised Weil numbers are roots of unity, which explains the observed periodicity.

In this paper we greatly extend the curve-based approach of [2], by developing an algorithm which for any prime power $q = p^r$, any positive integer $l < p$ and any integer $n \geq l$ coprime to p , outputs exact expressions in terms of the number of points over \mathbb{F}_{q^n} of certain affine varieties defined over \mathbb{F}_q , for all $F_q(n, t_1, \dots, t_l)$, and thus for all $I_q(n, t_1, \dots, t_l)$. If it is so desired then one can obtain explicit formulae for these numbers by computing the zeta functions of each of the arising varieties and combining the characteristic values in the appropriate way.

The ingredients of the algorithm we propose were initially developed for $q = 2$, since even the $l = 4$ cases were open problems with the only previous result being an approximation to the counts when $n \equiv 0, 2 \pmod{4}$ [22]. Were it not for the failure of Newton's identities in positive characteristic, our proposed algorithm would allow one to express the number of monic irreducibles in $\mathbb{F}_2[x]$ of odd degree in the above manner with *any* number of coefficients prescribed in *any* positions. Nevertheless, we give varieties (which all happen to be curves) for $l \leq 7$ and using Magma [4] compute the corresponding L-polynomials for $l = 4$ and $l = 5$, obtaining explicit formulae for these open problems for n odd. Interestingly, for these two cases the characteristic polynomial of the Frobenius endomorphism of the featured curves has factors that arise from ordinary – rather than supersingular – abelian varieties, and a simple argument implies that the set of formulae $\{F_2(n, t_1, t_2, t_3, t_4)\}_{n \geq 4}$ and $\{F_2(n, t_1, t_2, t_3, t_4, t_5)\}_{n \geq 5}$ for all fixed (t_1, \dots, t_4) and (t_1, \dots, t_5) respectively, can not be periodic in n , as in the $l = 2$ and $l = 3$ cases. This perhaps explains why there had been little progress in the four (or more) coefficients problem for the past 15 years, since there is not a finite set of cases to enumerate. The method for $q = 2$ extends to one for any $q = 2^r$ and $l \leq 7$, and also works for $q = 3^r$ and $l \leq 3$. In general, by restricting l to be strictly less than p one obviates the failure of Newton's identities and the algorithm provably performs as described for all $q = p^r$ and $l < p$ when $p \geq 5$.

In addition to the aforementioned transform of the problem of determining for a given $q = p^r$, $l < p$ and $n \geq l$ all $I_q(n, t_1, \dots, t_l)$, to the problem of determining all $F_q(n, t_1, \dots, t_l)$, the algorithm proceeds with two further transforms. The first transforms the latter problem to one of counting the number of evaluations to 1 of linear combinations of the trace functions. While this transform is valid for all prime powers, for $q = 2$ we use an interesting alternative which instead counts zeros and is related to Sylvester's construction of Hadamard matrices [33]. In order to count the number of evaluations to 1 (or zeros) of linear combinations of the trace functions, a third transform is used in order to express them as sums of the number of \mathbb{F}_{q^n} -rational points of certain affine varieties defined over \mathbb{F}_q . It is at this point that one requires that n be coprime to p . While this final transform is entirely elementary, determining what these varieties are appears to be a fundamentally computational problem, i.e., one should not expect to be able to simply write down general formulae for $F_q(n, t_1, \dots, t_l)$ for arbitrary q and $l < p$. There may of course exist faster algorithms for determining the relevant varieties and exact formulae, but we emphasise that the one presented here constitutes the first algorithmic approach to solving the prescribed coefficients problem exactly, which therefore represents a shift in perspective with regard to its study. Consequently, it is expected that by analysing the behaviour of the algorithm and the resulting varieties, further insight may be obtained into the prescribed coefficients problem.

The sequel is organised as follows. In §2 we detail our original algorithm for $q = 2$ and $l \leq 7$, as well as its possible limitations, while in §3 we apply the algorithm to give explicit formulae for $l = 4$ and $l = 5$, present the curves arising from the $l = 6$ and $l = 7$ cases, and detail a connection between the $l = 4$ case and binary Kloosterman sums. In §4 we present our main algorithm for arbitrary q and $l < p$, prove its correctness, and provide some examples of the arising varieties (again, all are curves) for $q = 5$. For good measure, in §5 we combine the algorithm for $q = 3$ with the method from §2 in order to give explicit formulae for $l \leq 3$. In §6 we make some final remarks and list some open problems and research directions.

We assume the reader is familiar with curves, abelian varieties, zeta functions and L-polynomials; should they not be, the relevant definitions may be found in [2, Section 3], for example. The code used for all interesting computations performed with Magma and Maple [1] is openly available from <https://github.com/robertgranger/CountingIrreducibles>, with the relevant files indicated in footnotes in the text.

2 The Motivating Case

In this section we present an algorithm for solving the prescribed coefficients problem exactly for $q = 2$, $l \leq 7$ and n odd. The $q = 2$ case is not only the simplest, but it is also very instructive since all of the key ideas behind the main algorithm presented in §4 are present. We now detail the two new problem transforms that are used, deferring what we refer to as Transform 1 – which expresses the $I_2(n, t_1, \dots, t_l)$ in terms of the $F_2(n, t_1, \dots, t_l)$ – to the appendix in the full version of this paper, since it is not new (see [6] or [22], for example).

2.1 Transform 2

We now transform the problem of counting field elements with prescribed traces to the problem of counting the number of zeros of linear combinations of the trace functions. We first fix some notation.

Let $f_0, \dots, f_{m-1} : \mathbb{F}_{2^n} \rightarrow \mathbb{F}_2$ be any functions and let $\mathbf{f} = (f_{m-1}, \dots, f_0)$. For $i, j \in \{0, \dots, 2^m - 1\}$ let $\mathbf{i} = (i_{m-1}, \dots, i_0)$ and $\mathbf{j} = (j_{m-1}, \dots, j_0)$ denote the binary expansions of i and j respectively, and let $\mathbf{i} \cdot \mathbf{j}$ denote their inner product mod 2. For any $i \in \{0, \dots, 2^m - 1\}$, let $\mathbf{i} \cdot \mathbf{f}$ denote the function

$$\sum_{k=0}^{m-1} i_k f_k : \mathbb{F}_{2^n} \rightarrow \mathbb{F}_2,$$

and let $Z(\mathbf{i} \cdot \mathbf{f})$ denote the number of zeros in \mathbb{F}_{2^n} of $\mathbf{i} \cdot \mathbf{f}$. We interpret $Z(\mathbf{0} \cdot \mathbf{f})$ to be the number of zeros of the empty function, which we define to be 2^n . Finally, let $N(\mathbf{j}) = N(j_{m-1}, \dots, j_0)$ denote the number of $a \in \mathbb{F}_{2^n}$ such that $f_k(a) = j_k$, for $k = 0, \dots, m-1$.

Our goal is to express any $N(\mathbf{j})$ in terms of the $Z(\mathbf{i} \cdot \mathbf{f})$, but we begin by first solving the inverse problem, i.e., expressing any $Z(\mathbf{i} \cdot \mathbf{f})$ in terms of the $N(\mathbf{j})$.

Lemma 1. *With the notation as above, for $0 \leq i \leq 2^m - 1$ we have:*

$$Z(\mathbf{i} \cdot \mathbf{f}) = \sum_{\mathbf{j}: \mathbf{j}=0} N(\mathbf{j}). \quad (2)$$

Proof. By definition, we have $Z(\mathbf{i} \cdot \mathbf{f}) = \#\{a \in \mathbb{F}_{2^n} \mid \mathbf{i} \cdot \mathbf{f}(a) = 0\} = \#\{a \in \mathbb{F}_{2^n} \mid \sum_{k=0}^{m-1} i_k f_k(a) = 0\}$. Since $N(\mathbf{j})$ counts precisely those $a \in \mathbb{F}_{2^n}$ such that $f_k(a) = j_k$, we must count over all those \mathbf{j} for which $\sum_{k=0}^{m-1} i_k j_k = 0$, i.e., those such that $\sum_{k=0}^{m-1} i_k j_k = 0$. \square

Writing Eq. (2) in matrix form, for $i, j \in \{0, \dots, 2^m - 1\}$ we have

$$[Z(\mathbf{i} \cdot \mathbf{f})]^T = S_m \cdot [N(\mathbf{j})]^T,$$

where $(S_m)_{i,j} = 1 - \mathbf{i} \cdot \mathbf{j}$ is nothing but Sylvester's construction [33] of Hadamard matrices [18], with the minus ones replaced with zeros. Now let H_m be the $2^m \times 2^m$ matrix with entries $(H_m)_{i,j} = (-1)^{\mathbf{i} \cdot \mathbf{j}}$, i.e., the scaled Walsh-Hadamard transform [37]. Since the Walsh-Hadamard matrix is involutory we have $H_m^{-1} = \frac{1}{2^m} H_m$. Also let A_m be the $2^m \times 2^m$ matrix with $(A_m)_{0,0} = 1$ and all other entries 0, and let B_m be the $2^m \times 2^m$ matrix with all entries 1. Lastly let Id_m be the $2^m \times 2^m$ identity matrix.

Lemma 2. *We have $S_m^{-1} = \frac{1}{2^{m-1}} H_m - A_m$.*

Proof. Noting that $2S_m = H_m + B_m$, we have

$$\begin{aligned} \left(\frac{1}{2^{m-1}} H_m - A_m \right) S_m &= \frac{1}{2^m} H_m \cdot 2S_m - A_m S_m \\ &= \frac{1}{2^m} H_m (H_m + B_m) - A_m S_m \\ &= \text{Id}_m + \frac{1}{2^m} H_m B_m - A_m S_m. \end{aligned}$$

Since all but the first row of H_m contains the same number of ones and minus ones, $\frac{1}{2^m} H_m B_m$ consists of the all-one vector in the first row and the all-zero vector for the others, as does $A_m S_m$. \square

Thus in order to compute any of the 2^m possible outputs $N(\mathbf{j})$ of any set of m functions \mathbf{f} , it is sufficient to count the number of zeros of all the 2^m \mathbb{F}_2 -linear combinations of the functions, and then apply S_m^{-1} . In particular, one may choose the f_{m-1}, \dots, f_0 to be any subset of the trace functions T_1, \dots, T_n . This transform is therefore far more general than is required for the target subproblems of interest.

2.2 Transform 3

We now transform the problem of counting the number of zeros of linear combinations of trace functions to the problem of counting the number of affine points on associated sets of varieties. As for Transform 2, for the sake of generality we allow arbitrary subsets of the trace functions to have prescribed values.

Let the input traces whose values are prescribed be $\mathbf{f} = (T_{l_{m-1}}, \dots, T_{l_0})$ with $l_{m-1} > \dots > l_0$. Then by Transform 2, for all $i \in \{1, \dots, 2^m - 1\}$ one needs to compute

$$Z(\mathbf{i} \cdot \mathbf{f}) = \#\{a \in \mathbb{F}_{2^n} \mid \sum_{k=0}^{m-1} i_k T_{l_k}(a) = 0\}. \quad (3)$$

In general this problem appears to be non-trivial, since the degree of each $T_{l_k}(a)$ in a and its Frobenius powers is l_k . However, it can be obviated – at least for n odd – by using the following degree-lowering idea.

Firstly, note that since the input to the trace functions has linear trace either 0 or 1, Eq. (3) can be rewritten as

$$Z(\mathbf{i} \cdot \mathbf{f}) = \#\{a \in \mathbb{F}_{2^n} \mid T_1(a) = 0, \sum_{k=0}^{m-1} i_k T_{l_k}(a) = 0\} + \#\{a \in \mathbb{F}_{2^n} \mid T_1(a) = 1, \sum_{k=0}^{m-1} i_k T_{l_k}(a) = 0\}.$$

Secondly, note that the condition $T_1(a) = 0$ is equivalent to $a = a_0^2 + a_0$, for two $a_0 \in \mathbb{F}_{2^n}$ (cf. [26, Theorem 2.25]), while for n odd the condition $T_1(a) = 1$ is equivalent to $a = a_0^2 + a_0 + 1$, for two $a_0 \in \mathbb{F}_{2^n}$. We therefore have

$$Z(\mathbf{i} \cdot \mathbf{f}) = \frac{1}{2} \sum_{r_0 \in \mathbb{F}_2} \#\{a_0 \in \mathbb{F}_{2^n} \mid \sum_{k=0}^{m-1} i_k T_{l_k}(a_0^2 + a_0 + r_0) = 0\}. \quad (4)$$

Thirdly, it happens that the functions $T_l(a_0^2 + a_0)$ and $T_l(a_0^2 + a_0 + 1)$ for $2 \leq l \leq 7$ are all expressible in characteristic two as polynomials of traces of lower degree whose arguments are polynomials in a_0 (see §2.2.1). Hence rather than having a single equation whose zeros one must count (Eq. (3)), one now has two equations whose number of zeros one must add and divide by 2 (Eq. (4)), both now of lower degree than before.

If after the above three steps there are terms that are not linear, i.e., not of the form $T_1(\cdot)$ for some argument, then the idea is to pick an argument of a trace function featuring in a non-linear term and apply the above three steps again. In particular, if the chosen argument is $g(a_0)$ then one introduces a new variable a_1 and as before writes $g(a_0) = a_1^2 + a_1 + r_1$ with $r_1 \in \mathbb{F}_2$ to account for whether the linear trace of $g(a_0)$ is 0 or 1, and expands all those terms in Eq. (4) which have this argument. This results in four equations whose number of zeros one must sum and divide by 4, with the degrees of the terms which feature this argument having been lowered, as before.

By recursively applying this idea and introducing variables a_0, \dots, a_{s-1} as necessary, with corresponding linear trace variables r_0, \dots, r_{s-1} , since the degrees of the non-linear terms always decreases one eventually obtains a set of 2^s trace equations of the form $T_1(g_{\mathbf{r}}(a_0, \dots, a_{s-1})) = 0$ indexed by $\mathbf{r} = (r_0, \dots, r_{s-1}) \in (\mathbb{F}_2)^s$, the vector of trace values of the s rewritten arguments. Each of these can be eliminated by introducing a final variable a_s and writing $a_s^2 + a_s = g_{\mathbf{r}}(a_0, \dots, a_{s-1})$. Together with the accompanying $s - 1$ equations for the rewritten arguments (the initial variable a having been completely eliminated in going from Eq. (3) to Eq. (4)), this gives a set of 2^s varieties whose number of \mathbb{F}_{2^n} -rational points must be summed and divided by 2^{s+1} in order to determine $Z(\mathbf{i} \cdot \mathbf{f})$. Note that as each variety is defined by s equations in the $s+1$ variables a_0, \dots, a_s , if these are complete intersections then the resulting varieties will be curves.

We now explain how to obtain expressions for $T_l(a_0^2 + a_0)$ and $T_l(a_0^2 + a_0 + 1)$ for $2 \leq l \leq 7$.

2.2.1 Computing $T_l(\alpha - \beta)$ Expressions for $T_2(\alpha - \beta)$ and $T_3(\alpha - \beta)$ in characteristic two were given by Fitzgerald and Yucas [12, Lemma 1.1] and proven by expanding the bilinear forms $T_l(\alpha + \beta) + T_l(\alpha) + T_l(\beta)$ in terms of the Frobenius powers of α and β , and deducing the correct function of the lower degree traces. It is possible – though laborious – to continue in this manner (we did so for $l = 4$), so instead we present an easier method.

We first recall Newton's identities over \mathbb{Z} (see e.g., [26, Theorem 1.75]) with indeterminates $\alpha_1, \dots, \alpha_n$. Abusing notation slightly, we refer to the elementary symmetric polynomials in $\alpha_1, \dots, \alpha_n$ as $T_1(\alpha), \dots, T_n(\alpha)$, and to the power sum symmetric polynomials $\alpha_1^k + \dots + \alpha_n^k$ as $T_1(\alpha^k)$ for $k \geq 1$, i.e., we work in the ring of symmetric functions, suppressing the dependence on n . We use the convention that $T_0(\alpha) = 1$.

Lemma 3. *For all $l \geq 1$ and $n \geq l$ we have*

$$l T_l(\alpha) = \sum_{k=1}^l (-1)^{k-1} T_{l-k}(\alpha) T_1(\alpha^k). \quad (5)$$

In order to use the argument $\alpha - \beta$ we need to work instead in the ring $\mathbb{Z}[\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n]$ and with the ring of multisymmetric functions in two variables, with the symmetric group S_n acting on $\alpha_1, \dots, \alpha_n$ and β_1, \dots, β_n independently (see [9] for a formal definition). Abusing notation slightly again, in this ring Eq. (5) becomes:

$$l T_l(\alpha - \beta) = \sum_{k=1}^l (-1)^{k-1} T_{l-k}(\alpha - \beta) T_1((\alpha - \beta)^k). \quad (6)$$

If one works over \mathbb{Q} rather than \mathbb{Z} then Eq. (6) leads to expressions for $T_l(\alpha - \beta)$ for any $l \geq 1$ as a sum of products of T_1 terms with arguments being various powers of $\alpha - \beta$. However, in characteristic two this is not so useful; even computing $T_2(\alpha - \beta)$ in this way is not possible as the l.h.s. vanishes. Nevertheless, working inductively for $2 \leq l \leq 7$ and applying Newton's identities evaluated at various products of powers of α and β so that no trace occurs to any power larger than one, all of the coefficients become divisible by l . Upon dividing by l one obtains an equation for $T_l(\alpha - \beta)$ over \mathbb{Z} , which can then be substituted into Eq. (6) in order to attempt to compute $T_{l+1}(\alpha - \beta)$. For example, from Eq. (6) we have

$$\begin{aligned} 2T_2(\alpha - \beta) &= T_1(\alpha - \beta)^2 - T_1((\alpha - \beta)^2) \\ &= (T_1(\alpha) - T_1(\beta))^2 - T_1(\alpha^2) + 2T_1(\alpha\beta) - T_1(\beta^2) \\ &= T_1(\alpha)^2 - 2T_1(\alpha)T_1(\beta) + T_1(\beta)^2 - T_1(\alpha^2) + 2T_1(\alpha\beta) - T_1(\beta^2) \\ &= 2T_2(\alpha) + 2T_2(\beta) - 2T_1(\alpha)T_1(\beta) + 2T_1(\alpha\beta), \end{aligned}$$

where in the final line we have used Eq. (5) for $l = 2$ for α and β separately. We have therefore proven that

$$T_2(\alpha - \beta) = T_2(\alpha) + T_2(\beta) - T_1(\alpha)T_1(\beta) + T_1(\alpha\beta).$$

The parts of the following lemma can either be proven by induction on n using the identity

$$T_l(\alpha_1, \dots, \alpha_n) = T_{l-1}(\alpha_1, \dots, \alpha_{n-1}) \alpha_n + T_l(\alpha_1, \dots, \alpha_{n-1}),$$

or via sequences of manipulations as described above[†].

Lemma 4. *For all $n \geq l$ we have:*

- (1) $T_1(\alpha - \beta) = T_1(\alpha) - T_1(\beta)$,
- (2) $T_2(\alpha - \beta) = T_2(\alpha) + T_2(\beta) - T_1(\alpha)T_1(\beta) + T_1(\alpha\beta)$,
- (3) $T_3(\alpha - \beta) = T_3(\alpha) - T_3(\beta) + T_1(\alpha)T_2(\beta) - T_1(\beta)T_2(\alpha) + T_1(\alpha)T_1(\alpha\beta) - T_1(\beta)T_1(\alpha\beta) + T_1(\alpha\beta^2) - T_1(\alpha^2\beta)$,
- (4) $T_4(\alpha - \beta) = T_4(\alpha) + T_4(\beta) - T_1(\alpha)T_3(\beta) - T_1(\beta)T_3(\alpha) + T_2(\alpha)T_2(\beta) - T_1(\alpha)T_1(\beta)T_1(\alpha\beta) + T_1(\alpha\beta)T_2(\alpha) + T_1(\alpha\beta)T_2(\beta) - T_1(\alpha)T_1(\alpha^2\beta) + T_1(\alpha)T_1(\alpha\beta^2) + T_1(\beta)T_1(\alpha^2\beta) - T_1(\beta)T_1(\alpha\beta^2) + T_1(\alpha^3\beta) - T_1(\alpha^2\beta^2) + T_1(\alpha\beta^3) + T_2(\alpha\beta)$,

[†]See `NewtonApproach_1_le_7.mw` for a derivation of these expressions.

$$\begin{aligned}
(5) \quad T_5(\alpha - \beta) &= T_5(\alpha) - T_5(\beta) + T_1(\alpha)T_4(\beta) - T_1(\beta)T_4(\alpha) + T_2(\beta)T_3(\alpha) - T_2(\alpha)T_3(\beta) + T_1(\alpha)T_1(\beta)T_1(\alpha^2\beta) \\
&\quad - T_1(\beta)T_1(\alpha\beta)T_2(\alpha) + T_1(\alpha)T_1(\alpha\beta)T_2(\beta) - T_1(\alpha)T_1(\beta)T_1(\alpha\beta^2) - T_1(\alpha^2\beta)T_2(\alpha) \\
&\quad - T_1(\alpha^2\beta)T_2(\beta) + T_1(\alpha\beta^2)T_2(\alpha) + T_1(\alpha\beta^2)T_2(\beta) + T_1(\alpha\beta)T_3(\alpha) - T_1(\alpha\beta)T_3(\beta) \\
&\quad - T_1(\alpha\beta)T_1(\alpha^2\beta) + T_1(\alpha\beta)T_1(\alpha\beta^2) + T_1(\alpha)T_1(\alpha^3\beta) - T_1(\alpha)T_1(\alpha^2\beta^2) + T_1(\alpha)T_1(\alpha\beta^3) \\
&\quad - T_1(\beta)T_1(\alpha^3\beta) + T_1(\beta)T_1(\alpha^2\beta^2) - T_1(\beta)T_1(\alpha\beta^3) + T_1(\alpha)T_2(\alpha\beta) - T_1(\beta)T_2(\alpha\beta) \\
&\quad - T_1(\alpha^4\beta) + 2T_1(\alpha^3\beta^2) - 2T_1(\alpha^2\beta^3) + T_1(\alpha\beta^4),
\end{aligned}$$

$$\begin{aligned}
(6) \quad T_6(\alpha - \beta) &= T_6(\alpha) + T_6(\beta) - T_1(\alpha)T_5(\beta) - T_1(\beta)T_5(\alpha) + T_2(\alpha)T_4(\beta) + T_2(\beta)T_4(\alpha) - T_3(\beta)T_3(\alpha) \\
&\quad + T_2(\alpha)T_2(\alpha\beta) + T_1(\alpha^5\beta) - 2T_1(\alpha^4\beta^2) + 2T_1(\alpha^3\beta^3) - 2T_1(\alpha^2\beta^4) + T_1(\alpha\beta^5) \\
&\quad + T_1(\alpha\beta)T_2(\alpha)T_2(\beta) + T_2(\alpha\beta)T_2(\beta) - T_1(\alpha)T_1(\alpha\beta)T_1(\alpha^2\beta) - T_1(\beta)T_1(\alpha\beta)T_1(\beta^2\alpha) \\
&\quad - T_1(\alpha)T_1(\beta)T_2(\alpha\beta) + T_1(\beta)T_1(\alpha^2\beta)T_2(\alpha) + T_1(\alpha)T_1(\alpha\beta^2)T_2(\beta) - T_1(\alpha)T_1(\beta)T_1(\alpha\beta^3) \\
&\quad + T_1(\alpha)T_1(\beta)T_1(\alpha^2\beta^2) + T_1(\alpha)T_1(\alpha\beta)T_1(\alpha\beta^2) - T_1(\beta)T_1(\alpha\beta^2)T_2(\alpha) - T_1(\alpha)T_1(\beta)T_1(\alpha^3\beta) \\
&\quad - T_1(\beta)T_1(\alpha\beta)T_3(\alpha) - T_1(\alpha)T_1(\alpha^2\beta)T_2(\beta) - T_1(\alpha)T_1(\alpha\beta)T_3(\beta) + T_1(\beta)T_1(\alpha\beta)T_1(\alpha^2\beta) \\
&\quad + 4T_3(\alpha\beta) + T_2(\alpha\beta^2) - T_1(\alpha)T_1(\alpha^4\beta) + 2T_1(\alpha)T_1(\alpha^3\beta^2) - 2T_1(\alpha)T_1(\alpha^2\beta^3) + T_1(\alpha)T_1(\alpha\beta^4) \\
&\quad + T_1(\beta)T_1(\alpha^4\beta) - 2T_1(\beta)T_1(\alpha^3\beta^2) + 2T_1(\beta)T_1(\alpha^2\beta^3) - T_1(\beta)T_1(\alpha\beta^4) + T_2(\alpha^2\beta) \\
&\quad + T_1(\alpha\beta)T_4(\alpha) + T_1(\alpha\beta)T_4(\beta) + T_1(\alpha\beta)T_1(\alpha^3\beta) + T_1(\alpha\beta)T_1(\alpha\beta^3) - T_1(\alpha\beta)T_2(\alpha\beta) \\
&\quad - T_1(\alpha^2\beta)T_3(\alpha) + T_1(\alpha^2\beta)T_3(\beta) - T_1(\alpha^2\beta)T_1(\alpha\beta^2) + T_1(\alpha\beta^2)T_3(\alpha) - T_1(\alpha\beta^2)T_3(\beta) \\
&\quad + T_1(\alpha^3\beta)T_2(\alpha) + T_1(\alpha^3\beta)T_2(\beta) - T_1(\alpha^2\beta^2)T_2(\alpha) - T_1(\alpha^2\beta^2)T_2(\beta) + T_1(\alpha\beta^3)T_2(\alpha) \\
&\quad + T_1(\alpha\beta^3)T_2(\beta).
\end{aligned}$$

$$\begin{aligned}
(7) \quad T_7(\alpha - \beta) &= T_7(\alpha) - T_7(\beta) + T_1(\alpha)T_6(\beta) - T_6(\alpha)T_1(\beta) - T_2(\alpha)T_5(\beta) + T_2(\beta)T_5(\alpha) + T_3(\alpha)T_4(\beta) \\
&\quad - T_3(\beta)T_4(\alpha) + T_1(\alpha\beta)T_5(\alpha) - T_1(\alpha\beta)T_5(\beta) + T_1(\alpha\beta^6) - T_1(\alpha^6\beta) - T_1(\alpha\beta)T_1(\alpha^4\beta) \\
&\quad + 2T_1(\alpha\beta)T_1(\alpha^3\beta^2) - 2T_1(\alpha\beta)T_1(\alpha^2\beta^3) + T_1(\alpha\beta)T_1(\alpha\beta^4) - T_1(\alpha^2\beta)T_1(\alpha^3\beta) \\
&\quad + T_1(\alpha^2\beta)T_1(\alpha^2\beta^2) - T_1(\alpha^2\beta)T_1(\alpha\beta^3) - T_2(\alpha\beta)T_3(\beta) - T_1(\beta)T_1(\alpha\beta^2)T_3(\alpha) \\
&\quad - T_1(\beta)T_1(\alpha^3\beta)T_2(\alpha) + T_1(\beta)T_1(\alpha^2\beta^2)T_2(\alpha) - T_1(\beta)T_2(\alpha\beta)T_2(\alpha) - T_1(\beta)T_1(\alpha\beta^3)T_2(\alpha) \\
&\quad + 3T_1(\alpha^5\beta^2) - 5T_1(\alpha^4\beta^3) + 5T_1(\alpha^3\beta^4) - 3T_1(\alpha^2\beta^5) + T_2(\alpha\beta)T_3(\alpha) + T_1(\alpha)T_1(\alpha\beta)T_1(\alpha^3\beta) \\
&\quad + 2T_1(\alpha)T_1(\alpha\beta)T_1(\alpha^2\beta^2) + T_1(\alpha)T_1(\alpha\beta)T_1(\alpha\beta^3) + T_1(\alpha)T_1(\alpha\beta)T_4(\beta) \\
&\quad - 3T_1(\alpha)T_1(\alpha\beta)T_2(\alpha\beta) + T_1(\alpha)T_1(\alpha^2\beta)T_3(\beta) - T_1(\alpha)T_1(\alpha^2\beta)T_1(\alpha\beta^2) \\
&\quad - T_1(\alpha)T_1(\alpha\beta^2)T_3(\beta) + T_1(\alpha)T_1(\alpha^3\beta)T_2(\beta) - T_1(\alpha)T_1(\alpha^2\beta^2)T_2(\beta) + T_1(\alpha)T_1(\beta)T_1(\alpha^4\beta) \\
&\quad - 2T_1(\alpha)T_1(\beta)T_1(\alpha^3\beta^2) + 2T_1(\alpha)T_1(\beta)T_1(\alpha^2\beta^3) - T_1(\alpha)T_1(\beta)T_1(\alpha\beta^4) \\
&\quad + T_1(\alpha)T_2(\alpha\beta)T_2(\beta) + T_1(\alpha)T_1(\alpha\beta^3)T_2(\beta) - T_1(\beta)T_1(\alpha\beta)T_1(\alpha^3\beta) \\
&\quad - 2T_1(\beta)T_1(\alpha\beta)T_1(\alpha^2\beta^2) - T_1(\beta)T_1(\alpha\beta)T_1(\alpha\beta^3) - T_1(\beta)T_1(\alpha\beta)T_4(\alpha) \\
&\quad + 3T_1(\beta)T_1(\alpha\beta)T_2(\alpha\beta) + T_1(\beta)T_1(\alpha^2\beta)T_3(\alpha) + T_1(\beta)T_1(\alpha^2\beta)T_1(\alpha\beta^2) \\
&\quad - T_1(\alpha^2\beta)T_2(\beta)T_2(\alpha) + T_1(\alpha\beta^2)T_2(\beta)T_2(\alpha) + T_1(\alpha\beta)T_2(\beta)T_3(\alpha) + T_1(\alpha)T_2(\alpha\beta^2) \\
&\quad + T_1(\alpha)T_2(\alpha^2\beta) + 10T_1(\alpha)T_3(\alpha\beta) - T_1(\beta)T_2(\alpha\beta^2) - T_1(\beta)T_2(\alpha^2\beta) - 10T_1(\beta)T_3(\alpha\beta) \\
&\quad + T_1(\alpha)T_1(\alpha^5\beta) - 2T_1(\alpha)T_1(\alpha^4\beta^2) - 2T_1(\alpha)T_1(\alpha^2\beta^4) + T_1(\alpha)T_1(\alpha\beta^5) - T_1(\beta)T_1(\alpha^5\beta) \\
&\quad + 2T_1(\beta)T_1(\alpha^4\beta^2) + 2T_1(\beta)T_1(\alpha^2\beta^4) - T_1(\beta)T_1(\alpha\beta^5) - T_1(\alpha^4\beta)T_2(\alpha) - T_1(\alpha^4\beta)T_2(\beta) \\
&\quad + 2T_1(\alpha^3\beta^2)T_2(\alpha) + 2T_1(\alpha^3\beta^2)T_2(\beta) - 2T_1(\alpha^2\beta^3)T_2(\alpha) - 2T_1(\alpha^2\beta^3)T_2(\beta) \\
&\quad + T_1(\alpha\beta^4)T_2(\alpha) + T_1(\alpha\beta^4)T_2(\beta) + T_1(\alpha^3\beta)T_3(\alpha) - T_1(\alpha^3\beta)T_3(\beta) - T_1(\alpha^2\beta^2)T_3(\alpha) \\
&\quad + T_1(\alpha^2\beta^2)T_3(\beta) + T_1(\alpha\beta^3)T_3(\alpha) - T_1(\alpha\beta^3)T_3(\beta) - T_1(\alpha^2\beta)T_4(\alpha) - T_1(\alpha^2\beta)T_4(\beta)
\end{aligned}$$

$$\begin{aligned}
& -T_1(\alpha^2\beta)T_2(\alpha\beta) + T_1(\alpha\beta^2)T_1(\alpha^3\beta) - T_1(\alpha\beta^2)T_1(\alpha^2\beta^2) + T_1(\alpha\beta^2)T_1(\alpha\beta^3) \\
& + T_1(\alpha\beta^2)T_4(\alpha) + T_1(\alpha\beta^2)T_4(\beta) + T_1(\alpha\beta^2)T_2(\alpha\beta) - T_1(\alpha\beta)T_1(\alpha^2\beta)T_2(\alpha) \\
& - T_1(\alpha\beta)T_1(\alpha^2\beta)T_2(\beta) + T_1(\alpha\beta)T_1(\alpha\beta^2)T_2(\alpha) + T_1(\alpha\beta)T_1(\alpha\beta^2)T_2(\beta) \\
& - T_1(\alpha\beta)T_2(\alpha)T_3(\beta) + T_1(\alpha)T_1(\beta)T_1(\alpha\beta)T_1(\alpha^2\beta) - T_1(\alpha)T_1(\beta)T_1(\alpha\beta)T_1(\alpha\beta^2).
\end{aligned}$$

Note that all of the terms appearing in Lemma 4 have total degree l , when one counts this by multiplying the degree l' of $T_{l'}$ and the degree of the argument in each trace and adds these numbers over each trace appearing in a given term. Hence when one reduces mod 2 and sets $\beta = \alpha^2$, the two terms $T_l(\alpha)$ and $T_l(\beta)$ cancel, leaving only T_l 's of degree $< l$, as claimed earlier.

Unfortunately, using Newton's identities evaluated for various l at products of powers of α and β so that no trace occurs to any power larger than one, fails for $l = 8$ due to the presence of the term $T_2(xy)^2$, which can not be eliminated while keeping the remaining terms' coefficients divisible by 8. Whether or not there exist such expressions for $T_l(\alpha - \beta)$ over \mathbb{Z} for $l \geq 8$, we leave as an open problem. Note that it is known that the ring of multisymmetric functions in two sets of variables $\alpha_1, \dots, \alpha_n$ and β_1, \dots, β_n is not generated over \mathbb{Z} by the elementary multisymmetric functions that we are using, unless $n = 2$ [9]. However, since we are only interested in a particular family of multisymmetric functions – namely $T_l(\alpha - \beta)$ – and not all of them, it is possible that such expressions exist.

3 Curves and Explicit Formulae for $q = 2$

In this section we detail how to determine the relevant curves for $l \leq 7$ and explicit formulae for $l \leq 5$, for n odd. In practice, rather than obtain a variety as sketched in the previous section for each $i \in \{0, \dots, 2^l - 1\}$, it is more efficient to compute a variety for each featured $T_{l_k}(a)$ and then combine them as appropriate according to whether i_k is 0 or 1 for a given i , as we do in the examples below.

The formulae arising from this approach for $l = 3$ were contained in [2, §§4&5] – although obtained in a slightly different manner – but we include them here for demonstration purposes and completeness. Note that once $Z(\mathbf{i} \cdot \mathbf{f})$ has been obtained for $\mathbf{f} = (T_l, \dots, T_1)$ and $i \in \{0, \dots, 2^l - 1\}$, these functions need not be recomputed for the subsequent $\mathbf{f} = (T_{l+1}, \dots, T_1)$. Also note that $T_l(1) = \binom{n}{l}$. For a polynomial $p \in \mathbb{Z}[X]$ let $\rho_n(p)$ denote the sum of the n -th powers of the (complex) roots of p .

3.1 Computing $F_2(n, t_1, t_2, t_3)$

Setting $\mathbf{f} = (T_3, T_2, T_1)$, by Transform 2 we have

$$\begin{bmatrix} F_2(n, 0, 0, 0) \\ F_2(n, 1, 0, 0) \\ F_2(n, 0, 1, 0) \\ F_2(n, 1, 1, 0) \\ F_2(n, 0, 0, 1) \\ F_2(n, 1, 0, 1) \\ F_2(n, 0, 1, 1) \\ F_2(n, 1, 1, 1) \end{bmatrix} = \begin{bmatrix} N(\mathbf{0}) \\ N(\mathbf{1}) \\ N(\mathbf{2}) \\ N(\mathbf{3}) \\ N(\mathbf{4}) \\ N(\mathbf{5}) \\ N(\mathbf{6}) \\ N(\mathbf{7}) \end{bmatrix} = \frac{1}{4} \begin{bmatrix} -3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} Z(\mathbf{0} \cdot \mathbf{f}) \\ Z(\mathbf{1} \cdot \mathbf{f}) \\ Z(\mathbf{2} \cdot \mathbf{f}) \\ Z(\mathbf{3} \cdot \mathbf{f}) \\ Z(\mathbf{4} \cdot \mathbf{f}) \\ Z(\mathbf{5} \cdot \mathbf{f}) \\ Z(\mathbf{6} \cdot \mathbf{f}) \\ Z(\mathbf{7} \cdot \mathbf{f}) \end{bmatrix}.$$

By definition we have $Z(\mathbf{0} \cdot \mathbf{f}) = 2^n$, while $Z(\mathbf{1} \cdot \mathbf{f}) = Z(T_1) = \#\{a \in \mathbb{F}_{2^n} \mid T_1(a) = 0\} = 2^{n-1}$. To determine $Z(\mathbf{i} \cdot \mathbf{f})$ for $2 \leq i \leq 7$, we use Lemma 4 parts (1) to (3). In particular, setting $\alpha = a_0^2$ and $\beta = a_0$ for $r_0 = 0$, and $\alpha = a_0^2 + a_0$ and $\beta = 1$ for $r_0 = 1$, and evaluating mod 2 gives the following:

$$\begin{aligned}
T_1(a_0^2 + a_0 + r_0) &= T_1(r_0), \\
T_2(a_0^2 + a_0 + r_0) &= T_1\left(a_0^3 + a_0 + r_0 \binom{n}{2}\right), \\
T_3(a_0^2 + a_0 + r_0) &= T_1\left(a_0^5 + a_0 + r_0 \left(a_0^3 + a_0 + \binom{n}{3}\right)\right).
\end{aligned}$$

For $2 \leq i \leq 7$ let $\mathbf{i} = (i_2, i_1, i_0)$. Applying Transform 3, the curves we are interested in for a of trace r_0 are

$$a_1^2 + a_1 = i_2 \left(a_0^5 + a_0 + r_0 \left(a_0^3 + a_0 + \binom{n}{3} \right) \right) + i_1 \left(a_0^3 + a_0 + r_0 \binom{n}{2} \right) + i_0 r_0. \quad (7)$$

These curves have genus 1 if $i_2 = 0$ and genus 2 if $i_2 = 1$, and are all supersingular. As pointed out in [2], this is why the formulae are periodic in n .

It is easy to prove that mod 2 the vector $(\binom{n}{3}, \binom{n}{2}, \binom{n}{1})$ is equal to $(0, 0, 1)$ if $n \equiv 1 \pmod{4}$, and $(1, 1, 1)$ if $n \equiv 3 \pmod{4}$. Hence there are two cases to consider when computing the L-polynomials of the curves specified in (7). In order to express $F_2(n, t_1, t_2, t_3)$ compactly, we define the following polynomials:

$$\begin{aligned} p_2 &= X^2 + 2X + 2, \\ p_4 &= X^4 + 2X^3 + 2X^2 + 4X + 4. \end{aligned}$$

Using Magma to compute the L-polynomials of the relevant curves and applying Transform 2 gives the following theorem.

Theorem 1. *For $n \geq 3$ we have*

$$\begin{aligned} F_2(n, 0, 0, 0) &= 2^{n-3} - \frac{1}{8} (8\rho_n(p_2) + 4\rho_n(p_4)) \quad \text{if } n \equiv 1, 3 \pmod{4} \\ F_2(n, 1, 0, 0) &= 2^{n-3} - \frac{1}{8} \cdot \begin{cases} 8\rho_n(p_2) + 4\rho_n(p_4) & \text{if } n \equiv 1 \pmod{4} \\ -4\rho_n(p_4) & \text{if } n \equiv 3 \pmod{4} \end{cases} \\ F_2(n, 0, 1, 0) &= 2^{n-3} - \frac{1}{8} (-4\rho_n(p_4)) \quad \text{if } n \equiv 1, 3 \pmod{4} \\ F_2(n, 1, 1, 0) &= 2^{n-3} - \frac{1}{8} \cdot \begin{cases} -8\rho_n(p_2) + 4\rho_n(p_4) & \text{if } n \equiv 1 \pmod{4} \\ -4\rho_n(p_4) & \text{if } n \equiv 3 \pmod{4} \end{cases} \\ F_2(n, 0, 0, 1) &= 2^{n-3} - \frac{1}{8} (-4\rho_n(p_4)) \quad \text{if } n \equiv 1, 3 \pmod{4} \\ F_2(n, 1, 0, 1) &= 2^{n-3} - \frac{1}{8} \cdot \begin{cases} -4\rho_n(p_4) & \text{if } n \equiv 1 \pmod{4} \\ -8\rho_n(p_2) + 4\rho_n(p_4) & \text{if } n \equiv 3 \pmod{4} \end{cases} \\ F_2(n, 0, 1, 1) &= 2^{n-3} - \frac{1}{8} (-8\rho_n(p_2) + 4\rho_n(p_4)) \quad \text{if } n \equiv 1, 3 \pmod{4} \\ F_2(n, 1, 1, 1) &= 2^{n-3} - \frac{1}{8} \cdot \begin{cases} -4\rho_n(p_4) & \text{if } n \equiv 1 \pmod{4} \\ 8\rho_n(p_2) + 4\rho_n(p_4) & \text{if } n \equiv 3 \pmod{4} \end{cases} \end{aligned}$$

Observe that $F_2(n, t_1, t_2)$ can be obtained similarly, or by adding $F_2(n, t_1, t_2, 0)$ and $F_2(n, t_1, t_2, 1)$ as given in Theorem 1. Likewise observe that $F_2(n, t_1)$ can be obtained as $F_2(n, t_1, 0, 0) + F_2(n, t_1, 0, 1) + F_2(n, t_1, 1, 0) + F_2(n, t_1, 1, 1)$, and that summing all the expressions gives 2^n , as expected.

3.2 Computing $F_2(n, t_1, t_2, t_3, t_4)$

Setting $\mathbf{f} = (T_4, T_3, T_2, T_1)$, by Transform 2 we have:

$$\begin{bmatrix} F_2(n, 0, 0, 0, 0) \\ F_2(n, 1, 0, 0, 0) \\ F_2(n, 0, 1, 0, 0) \\ F_2(n, 1, 1, 0, 0) \\ F_2(n, 0, 0, 1, 0) \\ F_2(n, 1, 0, 1, 0) \\ F_2(n, 0, 1, 1, 0) \\ F_2(n, 1, 1, 1, 0) \\ F_2(n, 0, 0, 0, 1) \\ F_2(n, 1, 0, 0, 1) \\ F_2(n, 0, 1, 0, 1) \\ F_2(n, 1, 1, 0, 1) \\ F_2(n, 0, 0, 1, 1) \\ F_2(n, 1, 0, 1, 1) \\ F_2(n, 0, 1, 1, 1) \\ F_2(n, 1, 1, 1, 1) \end{bmatrix} = \begin{bmatrix} N(0) \\ N(1) \\ N(2) \\ N(3) \\ N(4) \\ N(5) \\ N(6) \\ N(7) \\ N(8) \\ N(9) \\ N(10) \\ N(11) \\ N(12) \\ N(13) \\ N(14) \\ N(15) \end{bmatrix} = \frac{1}{8} \begin{bmatrix} -7 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} Z(0 \cdot \mathbf{f}) \\ Z(1 \cdot \mathbf{f}) \\ Z(2 \cdot \mathbf{f}) \\ Z(3 \cdot \mathbf{f}) \\ Z(4 \cdot \mathbf{f}) \\ Z(5 \cdot \mathbf{f}) \\ Z(6 \cdot \mathbf{f}) \\ Z(7 \cdot \mathbf{f}) \\ Z(8 \cdot \mathbf{f}) \\ Z(9 \cdot \mathbf{f}) \\ Z(10 \cdot \mathbf{f}) \\ Z(11 \cdot \mathbf{f}) \\ Z(12 \cdot \mathbf{f}) \\ Z(13 \cdot \mathbf{f}) \\ Z(14 \cdot \mathbf{f}) \\ Z(15 \cdot \mathbf{f}) \end{bmatrix}.$$

To determine $Z(\mathbf{i} \cdot \mathbf{f})$ for $8 \leq i \leq 15$, we use Lemma 4 part (4). In particular, setting $\alpha = a_0^2$ and $\beta = a_0$ for $r_0 = 0$, and $\alpha = a_0^2 + a_0$ and $\beta = 1$ for $r_0 = 1$, and evaluating mod 2 gives the following:

$$\begin{aligned} T_4(a_0^2 + a_0 + r_0) &= T_2(a_0^3) + T_2(a_0) + T_1(a_0^3)T_1(a_0) \\ &\quad + T_1\left(a_0^7 + a_0^5 + a_0^3 + r_0\left(a_0^3 + a_0 + (a_0^3 + a_0)\binom{n}{2} + \binom{n}{4}\right)\right). \end{aligned}$$

This can be reduced to a T_1 expression, or linearised, using the substitutions $a_0 = a_1^2 + a_1 + r_1$ and $a_0^3 = a_2^2 + a_2 + r_2$, where $r_1, r_2 \in \mathbb{F}_2$ are the traces of a_0 and a_0^3 respectively. This results in[†]

$$\begin{aligned} T_4(a_0^2 + a_0 + r_0) &= T_1\left(a_2^3 + a_2 + a_1^3 + a_1 + a_0^7 + a_0^5 + r_0 r_1 + r_0 r_2 + r_1 r_2 + r_2 \right. \\ &\quad \left. + (r_0 + 1)(r_1 + r_2)\binom{n}{2} + r_0\binom{n}{4}\right). \end{aligned}$$

For $8 \leq i \leq 15$ let $\mathbf{i} = (i_3, i_2, i_1, i_0)$. Applying Transform 3, the curves we are interested in are given by the following intersections:

$$\begin{aligned} a_3^2 + a_3 &= i_3\left(a_2^3 + a_2 + a_1^3 + a_1 + a_0^7 + a_0^5 + r_0 r_1 + r_0 r_2 + r_1 r_2 + r_2 + (r_0 + 1)(r_1 + r_2)\binom{n}{2} + r_0\binom{n}{4}\right) \\ &\quad + i_2\left(a_0^5 + a_0 + r_0\left(a_0^3 + a_0 + \binom{n}{3}\right)\right) + i_1\left(a_0^3 + a_0 + r_0\binom{n}{2}\right) + i_0 r_0, \\ a_0 &= a_1^2 + a_1 + r_1, \\ a_0^3 &= a_2^2 + a_2 + r_2. \end{aligned}$$

For $i_3 = 1$ the genus of all of these curves is 14. It is easy to prove that mod 2 one has

$$\left(\binom{n}{4}, \binom{n}{3}, \binom{n}{2}, \binom{n}{1}\right) \equiv \begin{cases} (0, 0, 0, 1) & \text{if } n \equiv 1 \pmod{8} \\ (0, 1, 1, 1) & \text{if } n \equiv 3 \pmod{8} \\ (1, 0, 0, 1) & \text{if } n \equiv 5 \pmod{8} \\ (1, 1, 1, 1) & \text{if } n \equiv 7 \pmod{8} \end{cases},$$

[†]See `NewtonApproach_1_le_7.mw` to verify the linearised expressions for 4, 5, 6 and 7 coefficients.

and hence there are four cases to consider when computing the L-polynomials of each of the above curves. In order to express $F_2(n, t_1, t_2, t_3, t_4)$ compactly, we further define the following polynomials:

$$\begin{aligned} p_{8,1} &= X^8 + 4X^7 + 6X^6 + 4X^5 + 2X^4 + 8X^3 + 24X^2 + 32X + 16, \\ p_{8,2} &= X^8 + 2X^6 + 4X^5 + 2X^4 + 8X^3 + 8X^2 + 16. \end{aligned}$$

Note that by [32, Proposition 1], both $p_{8,1}$ and $p_{8,2}$ are not the characteristic polynomials of the Frobenius endomorphism of supersingular abelian varieties. There are also two other polynomials which occur as factors of the L-polynomials of the above curves, but they are even polynomials, and hence can be ignored for n odd.

Using Magma to compute the L-polynomials of the relevant curves and applying Transform 2 gives the following theorem.

Theorem 2. *For odd $n > 4$ we have*

$$\begin{aligned} F_2(n, 0, 0, 0, 0) &= 2^{n-4} - \frac{1}{16} (4\rho_n(p_2) + \rho_n(p_4) + \rho_n(p_{8,1}) + \rho_n(p_{8,2})) \quad \text{if } n \equiv 1, 3, 5, 7 \pmod{8} \\ F_2(n, 1, 0, 0, 0) &= 2^{n-4} - \frac{1}{16} \cdot \begin{cases} 4\rho_n(p_2) + \rho_n(p_4) + \rho_n(p_{8,1}) + \rho_n(p_{8,2}) & \text{if } n \equiv 1 \pmod{8} \\ 2\rho_n(p_2) - \rho_n(p_4) - \rho_n(p_{8,1}) + \rho_n(p_{8,2}) & \text{if } n \equiv 3 \pmod{8} \\ \rho_n(p_4) - \rho_n(p_{8,1}) - \rho_n(p_{8,2}) & \text{if } n \equiv 5 \pmod{8} \\ -2\rho_n(p_2) - \rho_n(p_4) + \rho_n(p_{8,1}) - \rho_n(p_{8,2}) & \text{if } n \equiv 7 \pmod{8} \end{cases} \\ F_2(n, 0, 1, 0, 0) &= 2^{n-4} - \frac{1}{16} \cdot \begin{cases} -2\rho_n(p_2) - \rho_n(p_4) + \rho_n(p_{8,1}) - \rho_n(p_{8,2}) & \text{if } n \equiv 1, 5 \pmod{8} \\ 2\rho_n(p_2) - \rho_n(p_4) - \rho_n(p_{8,1}) + \rho_n(p_{8,2}) & \text{if } n \equiv 3, 7 \pmod{8} \end{cases} \\ F_2(n, 1, 1, 0, 0) &= 2^{n-4} - \frac{1}{16} \cdot \begin{cases} -4\rho_n(p_2) + \rho_n(p_4) + \rho_n(p_{8,1}) + \rho_n(p_{8,2}) & \text{if } n \equiv 1 \pmod{8} \\ -2\rho_n(p_2) - \rho_n(p_4) + \rho_n(p_{8,1}) - \rho_n(p_{8,2}) & \text{if } n \equiv 3 \pmod{8} \\ \rho_n(p_4) - \rho_n(p_{8,1}) - \rho_n(p_{8,2}) & \text{if } n \equiv 5 \pmod{8} \\ 2\rho_n(p_2) - \rho_n(p_4) - \rho_n(p_{8,1}) + \rho_n(p_{8,2}) & \text{if } n \equiv 7 \pmod{8} \end{cases} \\ F_2(n, 0, 0, 1, 0) &= 2^{n-4} - \frac{1}{16} (-2\rho_n(p_2) - \rho_n(p_4) + \rho_n(p_{8,1}) - \rho_n(p_{8,2})) \quad \text{if } n \equiv 1, 3, 5, 7 \pmod{8} \\ F_2(n, 1, 0, 1, 0) &= 2^{n-4} - \frac{1}{16} \cdot \begin{cases} -2\rho_n(p_2) - \rho_n(p_4) + \rho_n(p_{8,1}) - \rho_n(p_{8,2}) & \text{if } n \equiv 1 \pmod{8} \\ -4\rho_n(p_2) + \rho_n(p_4) + \rho_n(p_{8,1}) + \rho_n(p_{8,2}) & \text{if } n \equiv 3 \pmod{8} \\ 2\rho_n(p_2) - \rho_n(p_4) - \rho_n(p_{8,1}) + \rho_n(p_{8,2}) & \text{if } n \equiv 5 \pmod{8} \\ \rho_n(p_4) - \rho_n(p_{8,1}) - \rho_n(p_{8,2}) & \text{if } n \equiv 7 \pmod{8} \end{cases} \\ F_2(n, 0, 1, 1, 0) &= 2^{n-4} - \frac{1}{16} \cdot \begin{cases} \rho_n(p_4) - \rho_n(p_{8,1}) - \rho_n(p_{8,2}) & \text{if } n \equiv 1, 5 \pmod{8} \\ -4\rho_n(p_2) + \rho_n(p_4) + \rho_n(p_{8,1}) + \rho_n(p_{8,2}) & \text{if } n \equiv 3, 7 \pmod{8} \end{cases} \\ F_2(n, 1, 1, 1, 0) &= 2^{n-4} - \frac{1}{16} \cdot \begin{cases} 2\rho_n(p_2) - \rho_n(p_4) - \rho_n(p_{8,1}) + \rho_n(p_{8,2}) & \text{if } n \equiv 1 \pmod{8} \\ 4\rho_n(p_2) + \rho_n(p_4) + \rho_n(p_{8,1}) + \rho_n(p_{8,2}) & \text{if } n \equiv 3 \pmod{8} \\ -2\rho_n(p_2) - \rho_n(p_4) + \rho_n(p_{8,1}) - \rho_n(p_{8,2}) & \text{if } n \equiv 5 \pmod{8} \\ \rho_n(p_4) - \rho_n(p_{8,1}) - \rho_n(p_{8,2}) & \text{if } n \equiv 7 \pmod{8} \end{cases} \\ F_2(n, 0, 0, 0, 1) &= 2^{n-4} - \frac{1}{16} (\rho_n(p_4) - \rho_n(p_{8,1}) - \rho_n(p_{8,2})) \quad \text{if } n \equiv 1, 3, 5, 7 \pmod{8} \\ F_2(n, 1, 0, 0, 1) &= 2^{n-4} - \frac{1}{16} \cdot \begin{cases} \rho_n(p_4) - \rho_n(p_{8,1}) - \rho_n(p_{8,2}) & \text{if } n \equiv 1 \pmod{8} \\ -2\rho_n(p_2) - \rho_n(p_4) + \rho_n(p_{8,1}) - \rho_n(p_{8,2}) & \text{if } n \equiv 3 \pmod{8} \\ 4\rho_n(p_2) + \rho_n(p_4) + \rho_n(p_{8,1}) + \rho_n(p_{8,2}) & \text{if } n \equiv 5 \pmod{8} \\ 2\rho_n(p_2) - \rho_n(p_4) - \rho_n(p_{8,1}) + \rho_n(p_{8,2}) & \text{if } n \equiv 7 \pmod{8} \end{cases} \\ F_2(n, 0, 1, 0, 1) &= 2^{n-4} - \frac{1}{16} \cdot \begin{cases} 2\rho_n(p_2) - \rho_n(p_4) - \rho_n(p_{8,1}) + \rho_n(p_{8,2}) & \text{if } n \equiv 1, 5 \pmod{8} \\ -2\rho_n(p_2) - \rho_n(p_4) + \rho_n(p_{8,1}) - \rho_n(p_{8,2}) & \text{if } n \equiv 3, 7 \pmod{8} \end{cases} \end{aligned}$$

$$\begin{aligned}
F_2(n, 1, 1, 0, 1) &= 2^{n-4} - \frac{1}{16} \cdot \begin{cases} \rho_n(p_4) - \rho_n(p_{8,1}) - \rho_n(p_{8,2}) & \text{if } n \equiv 1 \pmod{8} \\ 2\rho_n(p_2) - \rho_n(p_4) - \rho_n(p_{8,1}) + \rho_n(p_{8,2}) & \text{if } n \equiv 3 \pmod{8} \\ -4\rho_n(p_2) + \rho_n(p_4) + \rho_n(p_{8,1}) + \rho_n(p_{8,2}) & \text{if } n \equiv 5 \pmod{8} \\ -2\rho_n(p_2) - \rho_n(p_4) + \rho_n(p_{8,1}) - \rho_n(p_{8,2}) & \text{if } n \equiv 7 \pmod{8} \end{cases} \\
F_2(n, 0, 0, 1, 1) &= 2^{n-4} - \frac{1}{16} (2\rho_n(p_2) - \rho_n(p_4) - \rho_n(p_{8,1}) + \rho_n(p_{8,2})) \quad \text{if } n \equiv 1, 3, 5, 7 \pmod{8} \\
F_2(n, 1, 0, 1, 1) &= 2^{n-4} - \frac{1}{16} \cdot \begin{cases} 2\rho_n(p_2) - \rho_n(p_4) - \rho_n(p_{8,1}) + \rho_n(p_{8,2}) & \text{if } n \equiv 1 \pmod{8} \\ \rho_n(p_4) - \rho_n(p_{8,1}) - \rho_n(p_{8,2}) & \text{if } n \equiv 3 \pmod{8} \\ -2\rho_n(p_2) - \rho_n(p_4) + \rho_n(p_{8,1}) - \rho_n(p_{8,2}) & \text{if } n \equiv 5 \pmod{8} \\ -4\rho_n(p_2) + \rho_n(p_4) + \rho_n(p_{8,1}) + \rho_n(p_{8,2}) & \text{if } n \equiv 7 \pmod{8} \end{cases} \\
F_2(n, 0, 1, 1, 1) &= 2^{n-4} - \frac{1}{16} \cdot \begin{cases} -4\rho_n(p_2) + \rho_n(p_4) + \rho_n(p_{8,1}) + \rho_n(p_{8,2}) & \text{if } n \equiv 1, 5 \pmod{8} \\ \rho_n(p_4) - \rho_n(p_{8,1}) - \rho_n(p_{8,2}) & \text{if } n \equiv 3, 7 \pmod{8} \end{cases} \\
F_2(n, 1, 1, 1, 1) &= 2^{n-4} - \frac{1}{16} \cdot \begin{cases} -2\rho_n(p_2) - \rho_n(p_4) + \rho_n(p_{8,1}) - \rho_n(p_{8,2}) & \text{if } n \equiv 1 \pmod{8} \\ \rho_n(p_4) - \rho_n(p_{8,1}) - \rho_n(p_{8,2}) & \text{if } n \equiv 3 \pmod{8} \\ 2\rho_n(p_2) - \rho_n(p_4) - \rho_n(p_{8,1}) + \rho_n(p_{8,2}) & \text{if } n \equiv 5 \pmod{8} \\ 4\rho_n(p_2) + \rho_n(p_4) + \rho_n(p_{8,1}) + \rho_n(p_{8,2}) & \text{if } n \equiv 7 \pmod{8} \end{cases}
\end{aligned}$$

One can check that the roots of $p_{8,1}$ are $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and their complex conjugates $\overline{\alpha_1}, \overline{\alpha_2}, \overline{\alpha_3}, \overline{\alpha_4}$, where:

$$\begin{aligned}
\alpha_1 &= -\frac{1}{2} + \frac{\sqrt{2}}{4} \left(1 + \sqrt{7 + 4\sqrt{2}} \right) - \frac{\sqrt{2}}{4} \left(1 + \sqrt{5 - 2\sqrt{2}} \right) i, \\
\alpha_2 &= -\frac{1}{2} + \frac{\sqrt{2}}{4} \left(1 - \sqrt{7 + 4\sqrt{2}} \right) - \frac{\sqrt{2}}{4} \left(1 - \sqrt{5 - 2\sqrt{2}} \right) i, \\
\alpha_3 &= -\frac{1}{2} - \frac{\sqrt{2}}{4} \left(1 + \frac{1}{\sqrt{17}} (3\sqrt{2} - 1) \sqrt{5 - 2\sqrt{2}} \right) + \frac{\sqrt{2}}{4} \left(1 - \frac{1}{\sqrt{17}} (3\sqrt{2} - 1) \sqrt{7 + 4\sqrt{2}} \right) i, \\
\alpha_4 &= -\frac{1}{2} - \frac{\sqrt{2}}{4} \left(1 - \frac{1}{\sqrt{17}} (3\sqrt{2} - 1) \sqrt{5 - 2\sqrt{2}} \right) + \frac{\sqrt{2}}{4} \left(1 + \frac{1}{\sqrt{17}} (3\sqrt{2} - 1) \sqrt{7 + 4\sqrt{2}} \right) i.
\end{aligned}$$

One can also check that the roots of $p_{8,2}$ are $i\alpha_1, i\alpha_2, i\alpha_3, i\alpha_4$ and $\overline{i\alpha_1}, \overline{i\alpha_2}, \overline{i\alpha_3}, \overline{i\alpha_4}$. In Theorem 2 the formulae for each $F_2(n, t_1, t_2, t_3, t_4)$ and each odd $n \pmod{8}$ have non-supersingular terms of the form $\pm \rho_n(p_{8,1}) \pm \rho_n(p_{8,2})$ or $\pm \rho_n(p_{8,1}) \mp \rho_n(p_{8,2})$. A simple application of Kronecker's theorem to the phases of these non-supersingular Weil numbers allows one to deduce that the formulae are not periodic in n .

3.2.1 Connection with binary Kloosterman sums The binary Kloosterman sum $\mathcal{K}_{2^n} : \mathbb{F}_{2^n} \rightarrow \mathbb{Z}$ can be defined by

$$\mathcal{K}_{2^n}(a) = 1 + \sum_{x \in \mathbb{F}_{2^n}^\times} (-1)^{T_1(x^{-1} + ax)}.$$

Kloosterman sums have applications in cryptography and coding theory, see for example [7,30]. In particular, zeros of \mathcal{K}_{2^n} lead to bent functions from $\mathbb{F}_{2^{2n}} \rightarrow \mathbb{F}_2$ [10]. The following elementary lemma connects Kloosterman sums to a family of elliptic curves.

Lemma 5 ([25]). *Let $a \in \mathbb{F}_{2^n}^\times$ and define the elliptic curve $E_{2^n}(a)$ over \mathbb{F}_{2^n} by*

$$E_{2^n}(a) : y^2 + xy = x^3 + a.$$

Then $\#E_{2^n}(a) = 2^n + \mathcal{K}_{2^n}(a)$.

Computing Kloosterman sum zeros is generally regarded as being difficult, currently taking exponential time (in n) to find a single non-trivial ($a \neq 0$) zero. Besides the deterministic test due to Ahmadi and Granger [3], which computes the cardinality of the Sylow 2-subgroup of any $E_{2^n}(a)$ via point-halving, and thus by Lemma 5 the maximum power of 2 dividing $\mathcal{K}_{2^n}(a)$, research has focused on characterising Kloosterman sums modulo small integers [29,27,13,19,28,16,17,15]. In order to analyse the expected running time of the algorithm of Ahmadi and Granger, it is necessary to know the distribution of Kloosterman sums which are divisible by successive powers of 2. Table 1 presents this distribution for $n \leq 13$, which was also presented in [3].

Let $T(n, k)$ denote the (n, k) -th entry of Table 1, i.e., the number of $a \in \mathbb{F}_{2^n}^\times$ for which $\#E_{2^n}(a)$ is divisible by 2^k . By using a result of Katz and Livné [20] it is possible to express $T(n, k)$ in terms of the class numbers of certain imaginary quadratic fields. However, it remains an open problem to give exact formulae for $k > 4$, with the formulae for the first four columns being as follows. Since the orders of all of the elliptic curves in Lemma 5 are divisible by 4, one has $T(n, 1) = T(n, 2) = 2^n - 1$. One can show that $E_{2^n}(a)$ has a point of order 8 if and only if $T_1(a) = 0$ (see e.g. [36]), hence $T_{2^n}(3) = 2^{n-1} - 1$. Finally, Lisoněk and Moiso proved that $T(n, 4) = (2^n - (-1 + i)^n - (-1 - i)^n)/4 - 1$, connecting it with the number of points on a supersingular elliptic curve [28, Theorem 3.6].

Table 1. $T(n, k) = \#\{a \in \mathbb{F}_{2^n}^\times \mid \#E_{2^n}(a) \equiv 0 \pmod{2^k}\}$

$n \backslash k$	1	2	3	4	5	6	7	8	9	10	11	12	13
1	1	1											
2	3	3											
3	7	7	3										
4	15	15	7	5									
5	31	31	15	5	5								
6	63	63	31	15	12	12							
7	127	127	63	35	14	14	14						
8	255	255	127	55	21	16	16	16					
9	511	511	255	135	63	18	18	18	18				
10	1023	1023	511	255	125	65	60	60	60	60			
11	2047	2047	1023	495	253	132	55	55	55	55	55		
12	4095	4095	2047	1055	495	252	84	72	72	72	72	72	
13	8191	8191	4095	2015	1027	481	247	52	52	52	52	52	52

The following theorem connects the distribution of binary Kloosterman sums mod 32 to the distribution of the first four coefficients of the characteristic polynomial.

Theorem 3 ([15]). *Let $a \in \mathbb{F}_{2^n}$ with $n \geq 4$ and let e_1, \dots, e_4 be the coefficients of the characteristic polynomial of a , regarded as integers. Then*

$$\mathcal{K}_{2^n}(a) \equiv 28e_1 + 8e_2 + 16(e_1e_2 + e_1e_3 + e_4) \pmod{32}.$$

Combining our Theorem 2 with Theorem 3 therefore provides explicit formulae for $\#\{a \in \mathbb{F}_{2^n} \mid \mathcal{K}_{2^n}(a) \equiv 0 \pmod{32}\} = T(n, 5) - 1$, as well as more generally the distribution of Kloosterman sums mod 32, for n odd. Indeed, this connection was our original motivation for considering the four coefficient problem. In particular, we have the following.

Corollary 1. *For odd $n > 4$ we have*

$$\#\{a \in \mathbb{F}_{2^n} \mid \mathcal{K}_{2^n}(a) \equiv 0 \pmod{32}\} = F_2(n, 0, 0, 0, 0) + F_2(n, 0, 0, 1, 0) = 2^{n-3} - \frac{1}{8}(\rho_n(p_2) + \rho_n(p_{8,1})).$$

a_0 and a_0^3 respectively. This results in

$$T_5(a_0^2 + a_0 + r_0) = T_1\left(r_0(a_2^3 + a_2 + a_1^3 + a_1) + a_0^9 + r_0a_0^7 + (r_1 + r_2)a_0^5 + r_1 + r_2 + r_1r_2 + r_0r_1r_2\right. \\ \left. + (r_0a_0^5 + r_0r_2)\binom{n}{2} + (r_0r_1 + r_0r_2)\binom{n}{3} + r_0\binom{n}{5}\right).$$

For $16 \leq i \leq 31$ let $\mathbf{i} = (i_4, i_3, i_2, i_1, i_0)$. Applying Transform 3, the curves we are interested in are given by the following intersections:

$$a_3^2 + a_3 = i_4\left(r_0(a_2^3 + a_2 + a_1^3 + a_1) + a_0^9 + r_0a_0^7 + (r_1 + r_2)a_0^5 + r_1 + r_2 + r_1r_2 + r_0r_1r_2\right. \\ \left. + (r_0a_0^5 + r_0r_2)\binom{n}{2} + (r_0r_1 + r_0r_2)\binom{n}{3} + r_0\binom{n}{5}\right) \\ + i_3\left(a_2^3 + a_2 + a_1^3 + a_1 + a_0^7 + a_0^5 + (r_0 + 1)(r_1 + r_2)\binom{n}{2} + r_0r_1 + r_0r_2 + r_1r_2 + r_2 + r_0\binom{n}{4}\right) \\ + i_2\left(a_0^5 + a_0 + r_0\left(a_0^3 + a_0 + \binom{n}{3}\right)\right) + i_1\left(a_0^3 + a_0 + r_0\binom{n}{2}\right) + i_0r_0, \\ a_0 = a_1^2 + a_1 + r_1, \\ a_0^3 = a_2^2 + a_2 + r_2.$$

For each $16 \leq i \leq 31$ the genus of all of these curves is 18. Again it is easy to prove that mod 2 one has

$$\left(\binom{n}{5}, \binom{n}{4}, \binom{n}{3}, \binom{n}{2}, \binom{n}{1}\right) \equiv \begin{cases} (0, 0, 0, 0, 1) & \text{if } n \equiv 1 \pmod{8} \\ (0, 0, 1, 1, 1) & \text{if } n \equiv 3 \pmod{8} \\ (1, 1, 0, 0, 1) & \text{if } n \equiv 5 \pmod{8} \\ (1, 1, 1, 1, 1) & \text{if } n \equiv 7 \pmod{8} \end{cases},$$

and hence there are four cases to consider when computing the L-polynomials of each of the above curves. In order to express $F_2(n, t_1, t_2, t_3, t_4, t_5)$ compactly, we define the following polynomial:

$$p_{8,3} = X^8 + 2X^7 + 2X^6 - 4X^4 + 8X^2 + 16X + 16.$$

As with the four coefficient case there are several other even polynomials which occur as factors of the L-polynomials of the above curves, which can hence be ignored for n odd.

We used Magma V22.2-3 to compute the L-polynomials of the relevant curves, which took just under 15 minutes on a 2GHz AMD Opteron computer. Applying Transform 2 gives the following theorem[†].

[†]See $F_2(\mathbf{n}, \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4, \mathbf{t}_5)$.m.

[illegible]

One can check that the roots $p_{8,3}$ are $\beta_1, \overline{\beta_1}, i\overline{\beta_1}, -i\beta_1$ and $\beta_2, \overline{\beta_2}, i\overline{\beta_2}, -i\beta_2$, where:

$$\begin{aligned}\beta_1 &= -\frac{1}{4}\left(1 - \sqrt{5} - \sqrt{10 + 2\sqrt{5}}\right) - \frac{1}{4}\left(1 - \sqrt{5} + \sqrt{10 + 2\sqrt{5}}\right)i, \\ \beta_2 &= -\frac{1}{8}\left(2(1 + \sqrt{5}) - (1 - \sqrt{5})\sqrt{10 + 2\sqrt{5}}\right) - \frac{1}{8}\left(2(1 + \sqrt{5}) + (1 - \sqrt{5})\sqrt{10 + 2\sqrt{5}}\right)i.\end{aligned}$$

Again, by applying Kronecker's theorem to the phases of the non-supersingular Weil numbers, one can deduce that the formulae are not periodic in n .

3.4 Computing $F_2(n, t_1, t_2, t_3, t_4, t_5, t_6)$

Let $\mathbf{f} = (T_6, T_5, T_4, T_3, T_2, T_1)$. To determine $Z(\mathbf{i} \cdot \mathbf{f})$ for $32 \leq i \leq 63$, we use Lemma 4 part (6). In particular, setting $\alpha = a_0^2$ and $\beta = a_0$ for $r_0 = 0$, and $\alpha = a_0^2 + a_0$ and $\beta = 1$ for $r_0 = 1$, and evaluating mod 2 gives the following:

$$\begin{aligned}T_6(a_0^2 + a_0 + r_0) &= T_3(a_0) + T_2(a_0^3)T_1(a_0^3) + T_2(a_0^3)T_1(a_0) + T_2(a_0)T_1(a_0^3) + T_2(a_0^5) + T_2(a_0) \\ &\quad + T_1(a_0^7)T_1(a_0^3) + T_1(a_0^7)T_1(a_0) + T_1(a_0^5)T_1(a_0^3) + T_1(a_0^3)T_1(a_0) \\ &\quad + r_0(T_2(a_0^3) + T_2(a_0) + T_1(a_0^3)T_1(a_0) + T_1(a_0^7))\binom{n}{2} \\ &\quad + T_1(a_0^{11} + a_0^7 + r_0(a_0^3 + a_0) + r_0(a_0^5 + a_0))\binom{n}{3} + r_0(a_0^3 + a_0)\binom{n}{4} + r_0\binom{n}{6}.\end{aligned}$$

This can be reduced to a T_1 expression using the substitutions $a_0 = a_1^2 + a_1 + r_1$, $a_0^3 = a_2^2 + a_2 + r_2$ and $a_0^5 = a_3^2 + a_3 + r_3$, where $r_1, r_2, r_3 \in \mathbb{F}_2$ are the traces of a_0, a_0^3 and a_0^5 respectively. This results in

$$\begin{aligned}T_6(a_0^2 + a_0 + r_0) &= T_1(a_3^3 + a_3 + (r_1 + r_2)(a_2^3 + a_2 + a_1^3 + a_1 + a_0^7) + a_1^5 + a_1^3 + a_0^{11} + a_0^7 + r_0r_1 + r_0r_2 \\ &\quad + r_1r_2 + r_2r_3 + (r_0(a_2^3 + a_2 + a_1^3 + a_1 + a_0^7 + r_1 + r_2 + r_1r_2 + 1) + r_1 + r_2 + r_3))\binom{n}{2} \\ &\quad + (r_0r_1 + r_0r_3 + r_1)\binom{n}{3} + r_0(r_1 + r_2)\binom{n}{4} + r_0\binom{n}{6}.\end{aligned}$$

For $32 \leq i \leq 63$ let $\mathbf{i} = (i_5, i_4, i_3, i_2, i_1, i_0)$. Applying Transform 3, the curves we are interested in are given by the following intersections:

$$\begin{aligned}a_4^2 + a_4 &= i_5(a_3^3 + a_3 + (r_1 + r_2)(a_2^3 + a_2 + a_1^3 + a_1 + a_0^7) + a_1^5 + a_1^3 + a_0^{11} + a_0^7 + r_0r_1 + r_0r_2 \\ &\quad + r_1r_2 + r_2r_3 + (r_0(a_2^3 + a_2 + a_1^3 + a_1 + a_0^7 + r_1 + r_2 + r_1r_2 + 1) + r_1 + r_2 + r_3))\binom{n}{2} \\ &\quad + (r_0r_1 + r_0r_3 + r_1)\binom{n}{3} + r_0(r_1 + r_2)\binom{n}{4} + r_0\binom{n}{6}) \\ &\quad + i_4(r_0(a_2^3 + a_2 + a_1^3 + a_1) + a_0^9 + r_0a_0^7 + (r_1 + r_2)a_0^5 + r_1 + r_2 + r_1r_2 + r_0r_1r_2 \\ &\quad + (r_0a_0^5 + r_0r_2)\binom{n}{2} + (r_0r_1 + r_0r_2)\binom{n}{3} + r_0\binom{n}{5}) \\ &\quad + i_3(a_2^3 + a_2 + a_1^3 + a_1 + a_0^7 + a_0^5 + (r_0 + 1)(r_1 + r_2)\binom{n}{2} + r_0r_1 + r_0r_2 + r_1r_2 + r_2 + r_0\binom{n}{4}) \\ &\quad + i_2(a_0^5 + a_0 + r_0(a_0^3 + a_0 + \binom{n}{3})) + i_1(a_0^3 + a_0 + r_0\binom{n}{2}) + i_0r_0,\end{aligned}$$

$$a_0 = a_1^2 + a_1 + r_1,$$

$$a_0^3 = a_2^2 + a_2 + r_2.$$

$$a_0^5 = a_3^2 + a_3 + r_3.$$

Again it is easy to prove that mod 2 one has

$$\left(\binom{n}{6}, \binom{n}{5}, \binom{n}{4}, \binom{n}{3}, \binom{n}{2}, \binom{n}{1}\right) \equiv \begin{cases} (0, 0, 0, 0, 0, 1) & \text{if } n \equiv 1 \pmod{8} \\ (0, 0, 0, 1, 1, 1) & \text{if } n \equiv 3 \pmod{8} \\ (0, 1, 1, 0, 0, 1) & \text{if } n \equiv 5 \pmod{8} \\ (1, 1, 1, 1, 1, 1) & \text{if } n \equiv 7 \pmod{8} \end{cases},$$

and hence there are four cases to consider when computing the L-polynomials of each of the above curves.

For $i_5 = 1$ the genus of all of the above curves is 50^\dagger . In order to compute their L-polynomials one can compute the number of \mathbb{F}_{2^n} -rational points of each for $1 \leq n \leq 50$, which is not a completely trivial matter. The state-of-the-art p -adic point counting algorithms of Tuitman [35,34] are unfortunately not easily adaptable to such intersections, while the prime 2 is problematic. A curve-specific analysis may yield the relevant L-polynomials more efficiently, but since our algorithm is arguably more interesting than the explicit formulae, we leave this as an open problem.

3.5 Computing $F_2(n, t_1, t_2, t_3, t_4, t_5, t_6, t_7)$

Let $\mathbf{f} = (T_7, T_6, T_5, T_4, T_3, T_2, T_1)$. To determine $Z(\mathbf{i} \cdot \mathbf{f})$ for $64 \leq i \leq 127$, we use Lemma 4 part (7). In particular, setting $\alpha = a_0^2$ and $\beta = a_0$ for $r_0 = 0$, and $\alpha = a_0^2 + a_0$ and $\beta = 1$ for $r_0 = 1$, and evaluating mod 2 gives the following:

$$\begin{aligned} T_7(a_0^2 + a_0 + r_0) &= r_0 T_3(a_0) + T_1(a_0^5) T_1(a_0^3) T_1(a_0) + r_0 (T_2(a_0^3) T_1(a_0^3) + T_2(a_0^3) T_1(a_0) + T_2(a_0) T_1(a_0^3)) \\ &\quad + T_2(a_0^3) T_1(a_0^5) + T_2(a_0^3) T_1(a_0) + T_2(a_0) T_1(a_0^5) + T_2(a_0) T_1(a_0) + r_0 (T_2(a_0) \\ &\quad + T_2(a_0^3)) \binom{n}{3} + r_0 T_2(a_0) + r_0 T_2(a_0^5) + T_1(a_0^9) T_1(a_0^3) + T_1(a_0^9) T_1(a_0) + T_1(a_0^7) T_1(a_0^5) \\ &\quad + r_0 T_1(a_0^7) T_1(a_0^3) + (r_0 + 1) T_1(a_0^7) T_1(a_0) + \left(r_0 + 1 + r_0 \binom{n}{2}\right) T_1(a_0^5) T_1(a_0^3) \\ &\quad + \left(1 + r_0 \binom{n}{2}\right) T_1(a_0^5) T_1(a_0) + \left(r_0 + 1 + r_0 \binom{n}{2} + r_0 \binom{n}{3}\right) T_1(a_0^3) T_1(a_0) \\ &\quad + T_1\left(a_0^{13} + (r_0 + 1) a_0^{11} + a_0^9 + r_0(a_0^7 + a_0^5 + a_0^3) + a_0 + r_0(a_0^9 + a_0^3 + a_0)\right) \binom{n}{2} \\ &\quad + r_0 a_0^7 \binom{n}{3} + r_0(a_0^5 + a_0) \binom{n}{4} + r_0(a_0^3 + a_0) \binom{n}{5} + r_0 \binom{n}{7}. \end{aligned}$$

As in the six coefficient case, this can be reduced to a T_1 expression using the substitutions $a_0 = a_1^2 + a_1 + r_1$, $a_0^3 = a_2^2 + a_2 + r_2$ and $a_0^5 = a_3^2 + a_3 + r_3$, where $r_1, r_2, r_3 \in \mathbb{F}_2$ are the traces of a_0, a_0^3 and a_0^5 respectively. This results in

$$\begin{aligned} T_7(a_0^2 + a_0 + r_0) &= T_1\left(r_0(a_3^3 + a_3) + (r_0 r_1 + r_0 r_2 + r_1 + r_3 + r_0 \binom{n}{3})(a_2^3 + a_2) + r_0(a_1^5 + a_1^3) \right. \\ &\quad \left. + (r_0 r_1 + r_0 r_2 + r_1 + r_3 + r_0 \binom{n}{3})(a_1^3 + a_1) + a_0^{13} + (r_0 + 1) a_0^{11} \right. \\ &\quad \left. + \left(r_1 + r_2 + 1 + r_0 \binom{n}{2}\right) a_0^9 + \left(r_0 + r_1 + r_3 + r_0 r_1 + r_0 r_2 + r_0 \binom{n}{3}\right) a_0^7 \right. \\ &\quad \left. + r_1 + r_0 r_2 + r_0 r_3 + r_1 r_2 + r_1 r_3 + r_2 r_3 + r_0 r_1 r_2 + r_0 r_2 r_3 + r_1 r_2 r_3 \right. \\ &\quad \left. + \left(r_1 + r_0 r_3 + r_1 r_2 + r_1 r_3 + r_2 r_3 + r_0 r_1 r_2 + r_0 r_1 r_3 + r_0 r_2 r_3\right) \binom{n}{2} \right) \end{aligned}$$

[†]See `6CoefficientsGenus.m`.

$$\begin{aligned}
& + \binom{n}{3} (r_0 r_1 + r_0 r_1 r_2) + \binom{n}{2} \binom{n}{3} (r_0 r_1 + r_0 r_2) + \binom{n}{4} (r_0 r_1 + r_0 r_3) \\
& + \binom{n}{5} (r_0 r_1 + r_0 r_2) + r_0 \binom{n}{7}.
\end{aligned}$$

For $64 \leq i \leq 127$ let $\mathbf{i} = (i_6, i_5, i_4, i_3, i_2, i_1, i_0)$. Applying Transform 3, the curves we are interested in are given by the following intersections:

$$\begin{aligned}
a_4^2 + a_4 &= i_6 \left(r_0 (a_3^3 + a_3) + (r_0 r_1 + r_0 r_2 + r_1 + r_3 + r_0 \binom{n}{3}) (a_2^3 + a_2) + r_0 (a_1^5 + a_1^3) \right. \\
&+ (r_0 r_1 + r_0 r_2 + r_1 + r_3 + r_0 \binom{n}{3}) (a_1^3 + a_1) + a_0^{13} + (r_0 + 1) a_0^{11} \\
&+ \left(r_1 + r_2 + 1 + r_0 \binom{n}{2} \right) a_0^9 + \left(r_0 + r_1 + r_3 + r_0 r_1 + r_0 r_2 + r_0 \binom{n}{3} \right) a_0^7 \\
&+ r_1 + r_0 r_2 + r_0 r_3 + r_1 r_2 + r_1 r_3 + r_2 r_3 + r_0 r_1 r_2 + r_0 r_2 r_3 + r_1 r_2 r_3 \\
&+ \left(r_1 + r_0 r_3 + r_1 r_2 + r_1 r_3 + r_2 r_3 + r_0 r_1 r_2 + r_0 r_1 r_3 + r_0 r_2 r_3 \right) \binom{n}{2} \\
&+ \binom{n}{3} (r_0 r_1 + r_0 r_1 r_2) + \binom{n}{2} \binom{n}{3} (r_0 r_1 + r_0 r_2) + \binom{n}{4} (r_0 r_1 + r_0 r_3) \\
&+ \binom{n}{5} (r_0 r_1 + r_0 r_2) + r_0 \binom{n}{7} \Big) \\
&+ i_5 \left(a_3^3 + a_3 + (r_1 + r_2) (a_2^3 + a_2 + a_1^3 + a_1 + a_0^7) + a_1^5 + a_1^3 + a_0^{11} + a_0^7 + r_0 r_1 + r_0 r_2 \right. \\
&+ r_1 r_2 + r_2 r_3 + \left(r_0 (a_2^3 + a_2 + a_1^3 + a_1 + a_0^7 + r_1 + r_2 + r_1 r_2 + 1) + r_1 + r_2 + r_3 \right) \binom{n}{2} \\
&+ (r_0 r_1 + r_0 r_3 + r_1) \binom{n}{3} + r_0 (r_1 + r_2) \binom{n}{4} + r_0 \binom{n}{6} \Big) \\
&+ i_4 \left(r_0 (a_2^3 + a_2 + a_1^3 + a_1) + a_0^9 + r_0 a_0^7 + (r_1 + r_2) a_0^5 + r_1 + r_2 + r_1 r_2 + r_0 r_1 r_2 \right. \\
&+ (r_0 a_0^5 + r_0 r_2) \binom{n}{2} + (r_0 r_1 + r_0 r_2) \binom{n}{3} + r_0 \binom{n}{5} \Big) \\
&+ i_3 \left(a_2^3 + a_2 + a_1^3 + a_1 + a_0^7 + a_0^5 + (r_0 + 1) (r_1 + r_2) \binom{n}{2} + r_0 r_1 + r_0 r_2 + r_1 r_2 + r_2 + r_0 \binom{n}{4} \right) \\
&+ i_2 \left(a_0^5 + a_0 + r_0 \left(a_0^3 + a_0 + \binom{n}{3} \right) \right) + i_1 \left(a_0^3 + a_0 + r_0 \binom{n}{2} \right) + i_0 r_0, \\
a_0 &= a_1^2 + a_1 + r_1, \\
a_0^3 &= a_2^2 + a_2 + r_2, \\
a_0^5 &= a_3^2 + a_3 + r_3.
\end{aligned}$$

Again it is easy to prove that mod 2 one has

$$\left(\binom{n}{7}, \binom{n}{6}, \binom{n}{5}, \binom{n}{4}, \binom{n}{3}, \binom{n}{2}, \binom{n}{1} \right) \equiv \begin{cases} (0, 0, 0, 0, 0, 0, 1) & \text{if } n \equiv 1 \pmod{8} \\ (0, 0, 0, 0, 1, 1, 1) & \text{if } n \equiv 3 \pmod{8} \\ (0, 0, 1, 1, 0, 0, 1) & \text{if } n \equiv 5 \pmod{8} \\ (1, 1, 1, 1, 1, 1, 1) & \text{if } n \equiv 7 \pmod{8} \end{cases},$$

and hence there are four cases to consider when computing the L-polynomials of each of the above curves.

For $i_6 = 1$, the genus of each of the above curves is 58^\dagger . Therefore, we again leave it as open problem to determine the L-polynomials of these curves.

Note that there are alternative ways to linearise the expressions for $T_l(a_0^2 + a_0 + r_0)$, which may result in curves of different genera. However, once the L-polynomials have been computed and Transform 2 is applied, they all must result in the same expressions for $F_2(n, t_1, \dots, t_l)$.

4 The Main Algorithm

In this section we present an algorithm for solving the prescribed coefficients problem exactly for any prime power $q = p^r$, any $l < p$ and any $n \geq l$ coprime to p . As in §2 we employ three transforms – also referred to as Transforms 1, 2 and 3 – again deferring the treatment of Transform 1, which expresses the $I_q(n, t_1, \dots, t_l)$ in terms of the $F_q(n, t_1, \dots, t_l)$, to the appendix in the full version of this paper.

4.1 Transform 2

We now transform the problem of counting the number of elements of \mathbb{F}_{q^n} with prescribed traces to the problem of counting the number of elements for which linear combinations of the trace functions evaluate to 1. We first fix some notation.

We require a bijection from the integers $\{0, \dots, q^m - 1\}$ to $(\mathbb{F}_q)^m$, the image of an input i being denoted by \mathbf{i} . One can for instance take the base- q expansion of i to give $i'_{m-1}q^{m-1} + \dots + i'_0$ and then set $\mathbf{i} = (i_{m-1}, \dots, i_0) = (\tau(i'_{m-1}), \dots, \tau(i'_0))$, where $\tau : \{0, \dots, q-1\} \rightarrow \mathbb{F}_q$ is defined by fixing a degree r monic irreducible $f \in \mathbb{F}_p[x]$ and a polynomial basis for $\mathbb{F}_{p^r}/\mathbb{F}_p$, and mapping the base- p expansion of an integer in $\{0, \dots, q-1\}$ to the polynomial with those coefficients. Note that according to this definition, $\mathbf{0} = (0, \dots, 0)$ is the all-zero vector in $(\mathbb{F}_q)^m$.

Let $f_0, \dots, f_{m-1} : \mathbb{F}_{q^n} \rightarrow \mathbb{F}_q$ be any functions and let $\mathbf{f} = (f_{m-1}, \dots, f_0)$. For $\mathbf{i} = (i_{m-1}, \dots, i_0), \mathbf{j} = (j_{m-1}, \dots, j_0) \in (\mathbb{F}_q)^m$ let $\mathbf{i} \cdot \mathbf{j}$ denote the usual inner product. For any $\mathbf{i} \in (\mathbb{F}_q)^m$, let $\mathbf{i} \cdot \mathbf{f}$ denote the function

$$\sum_{k=0}^{m-1} i_k f_k : \mathbb{F}_{q^n} \rightarrow \mathbb{F}_q.$$

As before let $N(\mathbf{j}) = N(j_{m-1}, \dots, j_0)$ denote the number of $a \in \mathbb{F}_{q^n}$ such that $f_k(a) = j_k$, for $k = 0, \dots, m-1$. Furthermore, let $Z_1(\mathbf{i} \cdot \mathbf{f})$ denote the number of elements of \mathbb{F}_{q^n} for which $\mathbf{i} \cdot \mathbf{f}$ evaluates to 1, and define $Z_1(\mathbf{0} \cdot \mathbf{f})$ to be q^n . The reason we normalise to 1 and avoid counting zeros is so that we avoid repeated relations (up to scalar multiples in \mathbb{F}_q^\times), which would mean that one could not express the q^m $N(\mathbf{j})$'s in terms of the q^m $Z_1(\mathbf{i} \cdot \mathbf{f})$'s by a linear transformation. As before we begin by first solving the inverse problem, i.e., expressing any $Z_1(\mathbf{i} \cdot \mathbf{f})$ in terms of the $N(\mathbf{j})$.

Lemma 6. *With the notation as above, for $\mathbf{i} \in (\mathbb{F}_q)^m \setminus \{\mathbf{0}\}$ we have:*

$$Z_1(\mathbf{i} \cdot \mathbf{f}) = \sum_{\mathbf{j} \cdot \mathbf{i} = 1} N(\mathbf{j}). \quad (8)$$

Proof. By definition, we have $Z_1(\mathbf{i} \cdot \mathbf{f}) = \#\{a \in \mathbb{F}_{q^n} \mid \mathbf{i} \cdot \mathbf{f}(a) = 1\} = \#\{a \in \mathbb{F}_{q^n} \mid \sum_{k=0}^{m-1} i_k f_k(a) = 1\}$. Since $N(\mathbf{j})$ counts precisely those $a \in \mathbb{F}_{q^n}$ such that $f_k(a) = j_k$, we must count over all those \mathbf{j} for which $\sum_{k=0}^{m-1} i_k j_k = 1$, i.e., those such that $\sum_{k=0}^{m-1} i_k j_k = 1$. \square

Writing Eq. (8) in matrix form, for $i, j \in \{0, \dots, q^m - 1\}$ we have

$$[Z_1(\mathbf{i} \cdot \mathbf{f})]^T = S_{q,m} \cdot [N(\mathbf{j})]^T,$$

[†]See `7CoefficientsGenus.m`.

where

$$(S_{q,m})_{i,j} = \begin{cases} 1 & \text{if } \mathbf{i} \cdot \mathbf{j} = 1 \text{ or if } \mathbf{i} = \mathbf{0} \\ 0 & \text{otherwise} \end{cases}. \quad (9)$$

We have the following lemma.

Lemma 7. *For all prime powers $q = p^r$ and $m \geq 1$, the $q^m \times q^m$ matrix $S_{q,m}$ is invertible over \mathbb{Q} .*

Proof. Indexing the rows and columns by i and j for $0 \leq i, j \leq q^m - 1$, the 0-th row of $S_{q,m}$ consists of 1's only, while besides the initial 1, the 0-th column consists of 0's only. Therefore no \mathbb{Q} -linear combination of rows 1 to $q^m - 1$ can cancel the 1 in position $(0, 0)$. Hence if one shows that the submatrix

$$S_{1 \leq i, j \leq q^m - 1} = \begin{cases} 1 & \text{if } \mathbf{i} \cdot \mathbf{j} = 1 \\ 0 & \text{otherwise} \end{cases} \quad (10)$$

of $S_{q,m}$ is invertible then we are done, since this implies that $S_{q,m}$ has full rank. We claim that the inverse of S is

$$S_{1 \leq i, j \leq q^m - 1}^{\text{inv}} = \frac{1}{q^{m-1}} \begin{cases} 1 & \text{if } \mathbf{i} \cdot \mathbf{j} = 1 \\ -1 & \text{if } \mathbf{i} \cdot \mathbf{j} = 0 \\ 0 & \text{otherwise} \end{cases}. \quad (11)$$

Let $R = S \cdot S^{\text{inv}}$. Then for $1 \leq i, j \leq q^m - 1$ one has

$$R_{i,j} = \sum_{k=1}^{q^m-1} S_{i,k} \cdot S_{k,j}^{\text{inv}} = \sum_{\substack{k=1, \\ \mathbf{i} \cdot \mathbf{k}=1, \mathbf{j} \cdot \mathbf{k}=1}}^{q^m-1} 1 - \sum_{\substack{k=1, \\ \mathbf{i} \cdot \mathbf{k}=1, \mathbf{j} \cdot \mathbf{k}=0}}^{q^m-1} 1. \quad (12)$$

If $i = j$ then the second term of the r.h.s. of Eq. (12) is zero, while the first is q^{m-1} , since if one chooses a non-zero component i_l of \mathbf{i} (of which there is at least one as $i \neq 0$), then one can freely choose the $m-1$ coefficients of \mathbf{k} other than k_l , while the condition $\mathbf{i} \cdot \mathbf{k} = 1$ entails that

$$k_l = (1 - \sum_{\substack{w=0, \\ w \neq l}}^{m-1} i_w k_w) / i_l,$$

which is well-defined because i_l is invertible.

Now assume $i \neq j$. If possible choose l, l' such that $l \neq l'$ and $i_l \neq 0$ and $j_{l'} \neq 0$. Then considering the set of all k for which $\mathbf{i} \cdot \mathbf{k} = 1$ as described above, as $k_{l'}$ varies over \mathbb{F}_q , so does $k_{l'} j_{l'}$. Hence both terms of the r.h.s. of Eq. (12) are precisely q^{m-2} , since there are $m-2$ free components of \mathbf{k} . For the remaining case where \mathbf{i} and \mathbf{j} both have only one non-zero component, in position l say, then both terms of the r.h.s. of Eq. (12) are zero, since for the first there is no k_l for which $i_l k_l = 1$ and $j_l k_l = 1$ since $i_l \neq j_l$, while for the second the condition $\mathbf{j} \cdot \mathbf{k} = 0$ implies $k_l = 0$, in which case $i_l k_l = 1$ can not hold. Therefore, R is the $q^m - 1 \times q^m - 1$ identity matrix. \square

To compute $N(\mathbf{0})$, we claim that

$$N(\mathbf{0}) = Z_1(\mathbf{0} \cdot \mathbf{f}) - \frac{1}{q^{m-1}} \sum_{i=1}^{q^m-1} Z_1(\mathbf{i} \cdot \mathbf{f}). \quad (13)$$

Since $Z_1(\mathbf{0} \cdot \mathbf{f}) = q^n = \sum_{j=0}^{q^m-1} N(\mathbf{j})$ by definition, Eq. (13) is equivalent to

$$\sum_{j=1}^{q^m-1} N(\mathbf{j}) = \frac{1}{q^{m-1}} \sum_{i=1}^{q^m-1} Z_1(\mathbf{i} \cdot \mathbf{f}) = \frac{1}{q^{m-1}} \sum_{i=1}^{q^m-1} \sum_{\mathbf{i} \cdot \mathbf{j}=1} N(\mathbf{j}), \quad (14)$$

with the latter equality given by Eq. (8). Considering the r.h.s. of Eq. (14), the number of occurrences of $N(\mathbf{j})$ is q^{m-1} since as we argued previously if one fixes an l for which $j_l \neq 0$, then one can choose the components of \mathbf{i} other than i_l freely, while i_l is then fixed by the condition $\mathbf{i} \cdot \mathbf{j} = 1$. We have therefore proven the following:

$$S_{q,m}^{-1} = \frac{1}{q^{m-1}} \left[\begin{array}{c|ccc} q^{m-1} & -1 & & \dots & -1 \\ \hline 0 & & & & \\ \vdots & & S_{1 \leq i,j \leq q^{m-1}}^{inv} & & \\ 0 & & & & \end{array} \right] = \begin{cases} 1 & \text{if } \mathbf{i} \cdot \mathbf{j} = 1 \\ -1 & \text{if } \mathbf{i} \cdot \mathbf{j} = 0 \\ 0 & \text{otherwise} \end{cases} \quad (15)$$

Thus in order to compute any of the q^m possible outputs $N(\mathbf{j})$ of any set of m functions \mathbf{f} , it is sufficient to count the number of evaluations to 1 of all the $q^m - 1$ non-zero \mathbb{F}_q -linear combinations of the functions, and then apply $S_{q,m}^{-1}$. In particular, one may choose the f_{m-1}, \dots, f_0 to be any subset of the trace functions T_1, \dots, T_n , or in our case of interest, any subset of the trace functions T_1, \dots, T_l .

4.2 Transform 3

We now transform the problem of counting the number of evaluations to 1 of linear combinations of trace functions to the problem of counting the number of affine points on associated sets of varieties.

Let the input traces whose values are prescribed be $\mathbf{f} = (T_{l_{m-1}}, \dots, T_{l_0})$ with $p > l_{m-1} > \dots > l_0 \geq 1$, and assume $(n, p) = 1$. By Transform 2, for all $i \in \{1, \dots, q^m - 1\}$ one needs to compute

$$Z_1(\mathbf{i} \cdot \mathbf{f}) = \#\{a \in \mathbb{F}_{q^n} \mid \sum_{k=0}^{m-1} i_k T_{l_k}(a) = 1\}. \quad (16)$$

For a fixed n for which $(n, p) = 1$, the map $\rho : \mathbb{F}_q \rightarrow \mathbb{F}_q : r_0 \mapsto nr_0$ is a permutation of \mathbb{F}_q , hence conditioning on the linear trace of a and observing that the condition $T_1(a) = nr_0$ is equivalent to $a = a_0^q - a_0 + r_0$, for q different $a_0 \in \mathbb{F}_{q^n}$ (cf. [26, Theorem 2.25]), we have

$$Z_1(\mathbf{i} \cdot \mathbf{f}) = \sum_{r_0 \in \mathbb{F}_q} \#\{a \in \mathbb{F}_{q^n} \mid T_1(a) = nr_0, \sum_{k=0}^{m-1} i_k T_{l_k}(a) = 1\} \quad (17)$$

$$= \frac{1}{q} \sum_{r_0 \in \mathbb{F}_q} \#\{a_0 \in \mathbb{F}_{q^n} \mid \sum_{k=0}^{m-1} i_k T_{l_k}(a_0^q - a_0 + r_0) = 1\}, \quad (18)$$

In order to evaluate the counts in Eq. (18) for each \mathbf{i} , we use a degree-lowering idea similar to the one described in §2.2, but which is much simpler in this scenario. Recall Eq (6):

$$l T_l(\alpha - \beta) = \sum_{k=1}^l (-1)^{k-1} T_{l-k}(\alpha - \beta) T_1((\alpha - \beta)^k).$$

By induction, each $T_l(\alpha - \beta)$ can be expressed as a multivariate polynomial in T_1 terms only whose arguments are monomials $\alpha^s \beta^t$, with the coefficient of each term having denominators at most $l!$. Since $l_{m-1} < p$ all of the coefficients are invertible mod p . We denote this T_1 -only form by $\overline{T}_l(\alpha - \beta)$.

Since by Eq. (18) one will set $\alpha = a_0^q - a_0, \beta = -r_0$ in $\overline{T}_{l_k}(\alpha - \beta)$, every term of the form $T_1(\alpha \beta^d)$ for $d \geq 0$ can immediately be set to zero since $T_1(\alpha \beta^d) = T_1((a_0^q - a_0)(-r_0)^d) = (-r_0)^d T_1(a_0^q - a_0) = 0$,

simplifying the expressions. For each $1 \leq i \leq q^m - 1$ one computes

$$L_{\mathbf{i}} = \sum_{k=0}^{m-1} i_k \overline{T}_{l_k} (a_0^q - a_0 + r_0), \quad (19)$$

which is a multivariate (generally non-linear) polynomial in T_1 terms whose arguments are powers of a_0 , and various constant terms depending on r_0 and n that can be written as a T_1 term via $c \mapsto T_1(c/n)$. Since one only needs to consider $n \bmod p$ in the expanded version of Eq. (19), we define this to be $\overline{n} \in \{1, \dots, p-1\}$ and will obtain a set of varieties for each such \overline{n} , which solve the problem for all \mathbb{F}_{q^n} such that $n \equiv \overline{n} \pmod{p}$.

The degree of any term of Eq. (19) which is a product of more than one T_1 term can be reduced by using substitutions of the form

$$a_0^{e_v} = a_v^q - a_v + r_v, \quad (20)$$

with $r_v \in \mathbb{F}_q$ so that $T_1(a_0^{e_v}) = \overline{n}r_v$. After some finite number $s-1$ such substitutions the resulting expression for $L_{\mathbf{i}}$ will have been completely linearised, leading to q^s trace equations of the form $T_1(h_{\mathbf{r}, \overline{n}}(a_0, \dots, a_{s-1})) = 1$, indexed by $\mathbf{r} = (r_0, \dots, r_{s-1}) \in (\mathbb{F}_q)^s$ which is $1/\overline{n}$ times the vector of trace values of the s rewritten arguments $a_0^{e_1}, \dots, a_0^{e_{s-1}}$. Each of these can be eliminated by introducing a final variable a_s and writing

$$a_s^q - a_s + \frac{1}{\overline{n}} = h_{\mathbf{r}, \overline{n}}(a_0, a_1, \dots, a_{s-1}).$$

Together with the accompanying $s-1$ equations for the rewritten arguments (the initial variable a having been completely eliminated in going from Eq. (16) to Eq. (18)), this gives a set of q^s varieties whose number of \mathbb{F}_{q^n} -rational points must be summed and divided by q^{s+1} in order to determine $Z_1(\mathbf{i} \cdot \mathbf{f})$, for $n \equiv \overline{n} \pmod{p}$. Again, as each variety is defined by s equations in the $s+1$ variables a_0, \dots, a_s , if they are complete intersections then the resulting varieties will be curves.

Transform 3 is summarised in Algorithm 1.

ALGORITHM 1: COMPUTE AFFINE VARIETIES FOR $Z_1(\mathbf{i} \cdot \mathbf{f})$

INPUT: $\mathbf{f} = (T_{l_{m-1}}, \dots, T_{l_0})$ with $p > l_{m-1} > \dots > l_0 \geq 1$, and $1 \leq i \leq q^m - 1$
OUTPUT: Varieties $\{V_{i, \mathbf{r}, \overline{n}}\}$ for $\mathbf{r} = (r_0, \dots, r_{s-1}) \in (\mathbb{F}_q)^s$ such that $\forall n \geq l_{m-1}$ for which $n \equiv \overline{n} \pmod{p}$, one has $Z_1(\mathbf{i} \cdot \mathbf{f}) = \frac{1}{q^s} \sum_{\mathbf{r} \in (\mathbb{F}_q)^s} \#V_{i, \mathbf{r}, \overline{n}}(\mathbb{F}_{q^n})$

1. Compute $\overline{T}_{l_k}(\alpha - \beta)$ for $k = 0, \dots, m-1$;
 2. For $1 \leq i \leq q^m - 1$ do:
 3. $L_{\mathbf{i}} \leftarrow \sum_{k=0}^{m-1} i_k \overline{T}_{l_k} (a_0^q - a_0 + r_0)$
 4. Linearise $L_{\mathbf{i}}$ using substitutions $a_0^{e_v} = a_v^q - a_v + r_v$ for $v = 1, \dots, s-1$
 5. Return $\{V_{i, \mathbf{r}, \overline{n}}\} = \{\{a_s^q - a_s + 1/\overline{n} = h_{\mathbf{r}, \overline{n}}(a_0, \dots, a_{s-1})\} \cup \{a_0^{e_v} = a_v^q - a_v + r_v\}_{1 \leq v \leq s-1}\}$
-

We remark that while Algorithm 1 is perfectly correct as stated, as in §3 it may be more efficient in practice to compute a variety for each featured T_{l_k} and then combine them per \mathbf{i} as appropriate, assuming that the eliminated variables are eliminated in a compatible manner, so that the set of varieties nest as one starts from T_{l_1} and sequentially incorporates $T_{l_2}, \dots, T_{l_{m-1}}$.

4.3 Some example varieties for $q = 5$

We now compute some curves associated to the functions $F_5(n, t_1, t_2, t_3, t_4)$ for $t_i \in \mathbb{F}_5$, $n > 5$ and $(n, 5) = 1$. We demonstrate that there are several ways to linearise a given expression, which lead to curves of different genera. However, as noted in §3.5, once the L-polynomials have been computed and Transform 2 is applied, they all must result in the same expressions for $F_q(n, t_1, \dots, t_l)$.

For $1 \leq l \leq 4$ the expressions (over \mathbb{Q}) for $\overline{T}_l(\alpha - \beta)$ are[†]:

$$\begin{aligned}
\overline{T}_1(\alpha - \beta) &= T_1(\alpha) - T_1(\beta) \\
\overline{T}_2(\alpha - \beta) &= \frac{1}{2}T_1(\alpha)^2 - T_1(\alpha)T_1(\beta) + \frac{1}{2}T_1(\beta)^2 - \frac{1}{2}T_1(\alpha^2) - \frac{1}{2}T_1(\beta^2) + T_1(\alpha\beta) \\
\overline{T}_3(\alpha - \beta) &= \frac{1}{6}T_1(\alpha)^3 - \frac{1}{2}T_1(\alpha)^2T_1(\beta) + \frac{1}{2}T_1(\alpha)T_1(\beta)^2 - \frac{1}{2}T_1(\alpha)T_1(\alpha^2) - \frac{1}{2}T_1(\alpha)T_1(\beta^2) \\
&\quad + T_1(\alpha)T_1(\alpha\beta) - \frac{1}{6}T_1(\beta)^3 + \frac{1}{2}T_1(\beta)T_1(\alpha^2) + \frac{1}{2}T_1(\beta)T_1(\beta^2) - T_1(\beta)T_1(\alpha\beta) + \frac{1}{3}T_1(\alpha^3) \\
&\quad - \frac{1}{3}T_1(\beta^3) - T_1(\alpha^2\beta) + T_1(\alpha\beta^2) \\
\overline{T}_4(\alpha - \beta) &= \frac{1}{24}T_1(\alpha)^4 + \frac{1}{24}T_1(\beta)^4 - \frac{1}{4}T_1(\alpha^4) - \frac{1}{4}T_1(\beta^4) + T_1(\alpha^3\beta) - \frac{3}{2}T_1(\alpha^2\beta^2) + T_1(\alpha\beta^3) \\
&\quad + \frac{1}{2}T_1(\alpha)T_1(\beta)T_1(\alpha^2) + \frac{1}{2}T_1(\alpha)T_1(\beta)T_1(\beta^2) - T_1(\alpha)T_1(\beta)T_1(\alpha\beta) + \frac{1}{8}T_1(\beta^2)^2 \\
&\quad - \frac{1}{2}T_1(\beta^2)T_1(\alpha\beta) + \frac{1}{2}T_1(\alpha\beta)^2 + \frac{1}{3}T_1(\alpha)T_1(\alpha^3) - \frac{1}{3}T_1(\alpha)T_1(\beta^3) - T_1(\alpha)T_1(\alpha^2\beta) \\
&\quad + T_1(\alpha)T_1(\alpha\beta^2) - \frac{1}{3}T_1(\beta)T_1(\alpha^3) + \frac{1}{3}T_1(\beta)T_1(\beta^3) + T_1(\beta)T_1(\alpha^2\beta) - T_1(\beta)T_1(\alpha\beta^2) \\
&\quad - \frac{1}{6}T_1(\alpha)^3T_1(\beta) + \frac{1}{4}T_1(\alpha)^2T_1(\beta)^2 - \frac{1}{4}T_1(\alpha)^2T_1(\alpha^2) - \frac{1}{4}T_1(\alpha)^2T_1(\beta^2) + \frac{1}{2}T_1(\alpha)^2T_1(\alpha\beta) \\
&\quad - \frac{1}{6}T_1(\alpha)T_1(\beta)^3 - \frac{1}{4}T_1(\beta)^2T_1(\alpha^2) - \frac{1}{4}T_1(\beta)^2T_1(\beta^2) + \frac{1}{2}T_1(\beta)^2T_1(\alpha\beta) + \frac{1}{8}T_1(\alpha^2)^2 \\
&\quad + \frac{1}{4}T_1(\alpha^2)T_1(\beta^2) - \frac{1}{2}T_1(\alpha^2)T_1(\alpha\beta).
\end{aligned}$$

Setting $\alpha = a_0^5 - a_0, \beta = -r_0$ and evaluating mod 5 gives:

$$\begin{aligned}
T_1(a_0^5 - a_0 + r_0) &= T_1(r_0), \\
T_2(a_0^5 - a_0 + r_0) &= T_1(a_0^6 + 4a_0^2 + r_0^2(3\overline{n} + 2)), \\
T_3(a_0^5 - a_0 + r_0) &= T_1(4a_0^{11} + a_0^7 + r_0(\overline{n} + 3)(a_0^6 - a_0^2) + r_0^3(\overline{n}^2 + 2\overline{n} + 2)), \\
T_4(a_0^5 - a_0 + r_0) &= 3T_1(a_0^2)^2 + 4T_1(a_0^2)T_1(a_0^6) + 3T_1(a_0^6)^2 + T_1(a_0^{16} + a_0^{12} + r_0(4\overline{n} + 3)a_0^{11} + a_0^8 \\
&\quad + r_0(\overline{n} + 2)a_0^7 + r_0^2(3\overline{n}^2 + 3)a_0^6 + 2a_0^4 + r_0^2(2\overline{n}^2 + 2)a_0^2 + r_0^4(4\overline{n}^3 + \overline{n}^2 + 4\overline{n} + 1)).
\end{aligned}$$

For $1 \leq i \leq 5^4 - 1$ let $\mathbf{i} = (i_3, i_2, i_1, i_0)$. For $i_3 = 0$ one immediately obtains curves without an intersection, while for $i_3 \neq 0$ applying Algorithm 1 produces the following set of curves of genus 802:

$$\begin{aligned}
a_3^5 - a_3 + 1/\overline{n} &= i_3(a_0^{16} + a_0^{12} + r_0(4\overline{n} + 3)a_0^{11} + a_0^8 + r_0(\overline{n} + 2)a_0^7 + 2a_0^4 + r_0^4(4\overline{n}^3 + \overline{n}^2 + 4\overline{n} + 1) \\
&\quad + \overline{n}^2r_0^2(2r_1 + 3r_2) + \overline{n}(3r_1^2 + 4r_1r_2 + 3r_2^2)) \\
&\quad + i_2(4a_0^{11} + a_0^7 + r_0(\overline{n} + 3)(a_0^6 - a_0^2) + r_0^3(\overline{n}^2 + 2\overline{n} + 2)) \\
&\quad + i_1(a_0^6 + 4a_0^2 + r_0^2(3\overline{n} + 2)) + i_0r_0, \\
a_1^5 - a_1 + r_1 &= a_0^2, \\
a_2^5 - a_2 + r_2 &= a_0^6.
\end{aligned}$$

[†]See [Section4.3.mw](#) for verification of all expressions appearing in this subsection.

As an alternative, note that the terms $3T_1(a_0^2)^2 + 4T_1(a_0^2)T_1(a_0^6) + 3T_1(a_0^6)^2 = \frac{1}{8}T_1((a_0^6 - a_0^2))^2$ arise from the term $\frac{1}{8}T_1(\alpha^2)^2$ of $\overline{T}_4(\alpha - \beta)$. Hence one can instead set $a_1^5 - a_1 + r_1 = a_0^5 - a_0^2$, leading to the following more manageable set of curves of genus 160 for $i_3 \neq 0$:

$$\begin{aligned}
a_2^5 - a_2 + 1/\bar{n} = & i_3 \left(a_0^{16} + a_0^{12} + a_0^8 + r_0(4\bar{n} + 3)a_0^{11} + r_0(\bar{n} + 2)a_0^7 + r_0^2(\bar{n}^2 + 1)(3a_0^6 + 2a_0^2) + 2a_0^4 \right. \\
& \left. + r_0^4(4\bar{n}^3 + \bar{n}^2 + 4\bar{n} + 1) + 3\bar{n}r_1^2 \right) \\
& + i_2 \left(4a_0^{11} + a_0^7 + r_0(\bar{n} + 3)(a_0^6 - a_0^2) + r_0^3(\bar{n}^2 + 2\bar{n} + 2) \right) \\
& + i_1 \left(a_0^6 + 4a_0^2 + r_0^2(3\bar{n} + 2) \right) + i_0 r_0, \\
a_1^5 - a_1 + r_1 = & a_0^6 - a_0^2.
\end{aligned}$$

Using substitutions of the latter type should in general produce simpler varieties; Algorithm 1 was stated in the form given purely for simplicity. While we have not attempted to compute the L-polynomials of these curves, it is clear that unless there is much cancellation or smoothness of the factors of the zeta functions occurring in the $Z_1(\mathbf{i} \cdot \mathbf{f})$, then the expressions for $F_q(n, t_1, \dots, t_l)$ will contain characteristic values of increasingly large degree as q and $l \rightarrow \infty$.

5 Curves and Explicit Formulae for $q = 3$

As a proof of concept example of Transform 2 from §4.1, in this section we detail how to determine the relevant curves and explicit formulae for $l = 3$ and $(n, 3) = 1$. We do not use Transform 3 from §4.2 since $\overline{T}_3(\alpha - \beta)$ is not computable mod 3, and so as in §3 we use Lemma 4 parts (1) to (3). Also as in §3, it is more efficient to compute linearised forms for $T_2(a)$ and $T_3(a)$ and to combine them as appropriate for each $i \in \{1, \dots, 3^3 - 1\}$. Note that for $r_0 \in \mathbb{F}_3$ one has $T_l(r_0) = \binom{n}{l} r_0$.

The reason for not computing formulae for four coefficients is that the terms $T_4(\alpha)$ and $T_4(\beta)$ in Lemma 4 part (4) do not cancel mod 3 and the method of introducing new variables in order to linearise $T_4(a_0^3 - a_0 + r_0)$ fails as a result.

5.1 Computing $F_3(n, t_1, t_2, t_3)$

Setting $\mathbf{f} = (T_3, T_2, T_1)$, by Transform 2 we have

[illegible]

By definition we have $Z_1(\mathbf{0} \cdot \mathbf{f}) = 3^n$, while $Z_1(\mathbf{1} \cdot \mathbf{f}) = Z_1(T_1) = \#\{a \in \mathbb{F}_{3^n} \mid T_1(a) = 1\} = 3^{n-1}$, as does $Z_1(\mathbf{2} \cdot \mathbf{f}) = Z_1(2T_1) = \#\{a \in \mathbb{F}_{3^n} \mid 2T_1(a) = 1\}$. To determine $Z_1(\mathbf{i} \cdot \mathbf{f})$ for $3 \leq i \leq 26$, setting

$\alpha = a_0^3 - a_0$ and $\beta = -r_0$, and using Lemma 4 parts (1) to (3) mod 3 gives the following[†]:

$$\begin{aligned} T_1(a_0^3 - a_0 + r_0) &= T_1(r_0), \\ T_2(a_0^3 - a_0 + r_0) &= T_1\left(a_0^4 - a_0^2 - r_0 \binom{n}{2}/n\right), \\ T_3(a_0^3 - a_0 + r_0) &= T_1\left(a_0^7 - a_0^5 + r_0(n+1)(a_0^4 - a_0^2) + r_0 \binom{n}{3}/n\right). \end{aligned}$$

For $3 \leq i \leq 26$ let $\mathbf{i} = (i_2, i_1, i_0)$. The curves we are interested in for a of trace r_0 are

$$a_1^3 - a_0 + 1/\bar{n} = i_2\left(a_0^7 - a_0^5 - r_0(\bar{n}+1)(a_0^4 - a_0^2) - r_0 \binom{\bar{n}}{3}/\bar{n}\right) + i_1\left(a_0^4 - a_0^2 + r_0 \binom{\bar{n}}{2}/\bar{n}\right) + i_0 r_0.$$

These curves have genus 3 if $i_2 = 0$ and genus 6 if $i_2 \neq 0$. Again it is easy to prove that mod 3 one has

$$\left(\binom{n}{3}, \binom{n}{2}, \binom{n}{1}\right) \equiv \begin{cases} (0, 0, 1) & \text{if } n \equiv 1 \pmod{9} \\ (0, 1, -1) & \text{if } n \equiv 2 \pmod{9} \\ (1, 0, 1) & \text{if } n \equiv 4 \pmod{9} \\ (1, 1, -1) & \text{if } n \equiv 5 \pmod{9} \\ (-1, 0, 1) & \text{if } n \equiv 7 \pmod{9} \\ (-1, 1, -1) & \text{if } n \equiv 8 \pmod{9} \end{cases},$$

and hence there are six cases to consider when computing the L-polynomials of each of the above curves. In order to express $F_3(n, t_1, t_2, t_3)$ compactly, we define the following eight polynomials:

$$\begin{aligned} p_{2,1} &= X^2 - 3X + 3, \\ p_{2,2} &= X^2 + 3X + 3, \\ p_{2,3} &= X^2 + 3, \\ p_6 &= X^6 + 3X^5 + 9X^4 + 15X^3 + 27X^2 + 27X + 27, \\ p_{12,1} &= X^{12} - 3X^{11} + 3X^9 + 9X^8 - 45X^6 + 81X^4 + 81X^3 - 729X + 729, \\ p_{12,2} &= X^{12} - 3X^{11} + 12X^9 - 18X^8 - 27X^7 + 117X^6 - 81X^5 - 162X^4 + 324X^3 - 729X + 729, \\ p_{12,3} &= X^{12} - 3X^{11} + 9X^{10} - 15X^9 + 36X^8 - 54X^7 + 117X^6 - 162X^5 + 324X^4 - 405X^3 + 729X^2 \\ &\quad - 729X + 729, \\ p_{12,4} &= X^{12} + 6X^{11} + 18X^{10} + 39X^9 + 63X^8 + 81X^7 + 117X^6 + 243X^5 + 567X^4 + 1053X^3 + 1458X^2 \\ &\quad + 1458X + 729. \end{aligned}$$

Observe that by [32, Proposition 1], $p_6, p_{12,2}, p_{12,3}$ and $p_{12,4}$ are not the characteristic polynomials of the Frobenius endomorphism of supersingular abelian varieties and thus their roots are not all roots of unity. Furthermore, all polynomials (including $p_{12,1}$) have a solvable Galois group.

Using Magma to compute the L-polynomials of the relevant curves and applying Transform 2 gives the following theorem[‡], where $\mathbf{v} = (\rho_n(p_{2,1}), \rho_n(p_{2,2}), \rho_n(p_{2,3}), \rho_n(p_6), \rho_n(p_{12,1}), \rho_n(p_{12,2}), \rho_n(p_{12,3}), \rho_n(p_{12,4}))$.

[†]See `Ternary3Coefficients.mw`.

[‡]See `F3(n,t1,t2,t3).m` for generation and counting code, and `F3(n,t1,t2,t3)verify.mw` for verification of $F_3(n, 0, 0, 0)$, which can easily be adapted for the other cases.

Theorem 5. For $n \geq 3$ we have

$$\begin{aligned}
F_3(n, 0, 0, 0) &= 3^{n-3} - \frac{1}{81}(-21, -15, -18, -8, -14, -14, -8) \cdot \mathbf{v} \quad \text{if } n \equiv 1, 2, 4, 5, 7, 8 \pmod{9} \\
F_3(n, 1, 0, 0) &= 3^{n-3} - \frac{1}{81} \cdot \begin{cases} (-3, 3, 0, 4, -2, -2, -2, 4) \cdot \mathbf{v} & \text{if } n \equiv 1 \pmod{9} \\ (3, 0, -3, 4, -2, 4, -2, -2) \cdot \mathbf{v} & \text{if } n \equiv 2 \pmod{9} \\ (-3, 3, 0, -2, 1, 1, 1, -2) \cdot \mathbf{v} & \text{if } n \equiv 4, 7 \pmod{9} \\ (3, 0, -3, -2, 1, -2, 1, 1) \cdot \mathbf{v} & \text{if } n \equiv 5, 8 \pmod{9} \end{cases} \\
F_3(n, 2, 0, 0) &= 3^{n-3} - \frac{1}{81} \cdot \begin{cases} (-3, 3, 0, 4, -2, -2, -2, 4) \cdot \mathbf{v} & \text{if } n \equiv 1 \pmod{9} \\ (0, -3, 3, 4, -2, -2, 4, -2) \cdot \mathbf{v} & \text{if } n \equiv 2 \pmod{9} \\ (-3, 3, 0, -2, 1, 1, 1, -2) \cdot \mathbf{v} & \text{if } n \equiv 4, 7 \pmod{9} \\ (0, -3, 3, -2, 1, 1, -2, 1) \cdot \mathbf{v} & \text{if } n \equiv 5, 8 \pmod{9} \end{cases} \\
F_3(n, 0, 1, 0) &= 3^{n-3} - \frac{1}{81} \cdot \begin{cases} (12, 9, 6, 4, -2, 4, -2, -2) \cdot \mathbf{v} & \text{if } n \equiv 1, 4, 7 \pmod{9} \\ (9, 6, 12, -8, -14, 4, 10, 4) \cdot \mathbf{v} & \text{if } n \equiv 2, 5, 8 \pmod{9} \end{cases} \\
F_3(n, 1, 1, 0) &= 3^{n-3} - \frac{1}{81} \cdot \begin{cases} (3, 0, -3, -2, 1, -2, 1, 1) \cdot \mathbf{v} & \text{if } n \equiv 1, 7 \pmod{9} \\ (-3, 3, 0, 4, -2, -2, -2, 4) \cdot \mathbf{v} & \text{if } n \equiv 2 \pmod{9} \\ (3, 0, -3, 4, -2, 4, -2, -2) \cdot \mathbf{v} & \text{if } n \equiv 4 \pmod{9} \\ (-3, 3, 0, -2, 1, 1, 1, -2) \cdot \mathbf{v} & \text{if } n \equiv 5, 8 \pmod{9} \end{cases} \\
F_3(n, 2, 1, 0) &= 3^{n-3} - \frac{1}{81} \cdot \begin{cases} (3, 0, -3, -2, 1, -2, 1, 1) \cdot \mathbf{v} & \text{if } n \equiv 1, 5, 7, 8 \pmod{9} \\ (3, 0, -3, 4, -2, 4, -2, -2) \cdot \mathbf{v} & \text{if } n \equiv 2, 4 \pmod{9} \end{cases} \\
F_3(n, 0, 2, 0) &= 3^{n-3} - \frac{1}{81} \cdot \begin{cases} (9, 6, 12, 4, -2, -2, 4, -2) \cdot \mathbf{v} & \text{if } n \equiv 1, 4, 7 \pmod{9} \\ (12, 9, 6, -8, -14, 10, 4, 4) \cdot \mathbf{v} & \text{if } n \equiv 2, 5, 8 \pmod{9} \end{cases} \\
F_3(n, 1, 2, 0) &= 3^{n-3} - \frac{1}{81} \cdot \begin{cases} (0, -3, 3, -2, 1, 1, -2, 1) \cdot \mathbf{v} & \text{if } n \equiv 1, 4, 5, 8 \pmod{9} \\ (0, -3, 3, 4, -2, -2, 4, -2) \cdot \mathbf{v} & \text{if } n \equiv 2, 7 \pmod{9} \end{cases} \\
F_3(n, 2, 2, 0) &= 3^{n-3} - \frac{1}{81} \cdot \begin{cases} (0, -3, 3, -2, 1, 1, -2, 1) \cdot \mathbf{v} & \text{if } n \equiv 1, 4 \pmod{9} \\ (-3, 3, 0, 4, -2, -2, -2, 4) \cdot \mathbf{v} & \text{if } n \equiv 2 \pmod{9} \\ (-3, 3, 0, -2, 1, 1, 1, -2) \cdot \mathbf{v} & \text{if } n \equiv 5, 8 \pmod{9} \\ (0, -3, 3, 4, -2, -2, 4, -2) \cdot \mathbf{v} & \text{if } n \equiv 7 \pmod{9} \end{cases} \\
F_3(n, 0, 0, 1) &= 3^{n-3} - \frac{1}{81}(-21, -15, -18, 4, 7, 7, 4) \cdot \mathbf{v} \quad \text{if } n \equiv 1, 2, 4, 5, 7, 8 \pmod{9} \\
F_3(n, 1, 0, 1) &= 3^{n-3} - \frac{1}{81} \cdot \begin{cases} (-3, 3, 0, -2, 1, 1, 1, -2) \cdot \mathbf{v} & \text{if } n \equiv 1, 7 \pmod{9} \\ (3, 0, -3, -2, 1, -2, 1, 1) \cdot \mathbf{v} & \text{if } n \equiv 2, 5 \pmod{9} \\ (-3, 3, 0, 4, -2, -2, -2, 4) \cdot \mathbf{v} & \text{if } n \equiv 4 \pmod{9} \\ (3, 0, -3, 4, -2, 4, -2, -2) \cdot \mathbf{v} & \text{if } n \equiv 8 \pmod{9} \end{cases} \\
F_3(n, 2, 0, 1) &= 3^{n-3} - \frac{1}{81} \cdot \begin{cases} (-3, 3, 0, -2, 1, 1, 1, -2) \cdot \mathbf{v} & \text{if } n \equiv 1, 4 \pmod{9} \\ (0, -3, 3, -2, 1, 1, -2, 1) \cdot \mathbf{v} & \text{if } n \equiv 2, 8 \pmod{9} \\ (0, -3, 3, 4, -2, -2, 4, -2) \cdot \mathbf{v} & \text{if } n \equiv 5 \pmod{9} \\ (-3, 3, 0, 4, -2, -2, -2, 4) \cdot \mathbf{v} & \text{if } n \equiv 7 \pmod{9} \end{cases} \\
F_3(n, 0, 1, 1) &= 3^{n-3} - \frac{1}{81} \cdot \begin{cases} (12, 9, 6, -2, 1, -2, 1, 1) \cdot \mathbf{v} & \text{if } n \equiv 1, 4, 7 \pmod{9} \\ (9, 6, 12, 4, 7, -2, -5, -2) \cdot \mathbf{v} & \text{if } n \equiv 2, 5, 8 \pmod{9} \end{cases} \\
F_3(n, 1, 1, 1) &= 3^{n-3} - \frac{1}{81} \cdot \begin{cases} (3, 0, -3, -2, 1, -2, 1, 1) \cdot \mathbf{v} & \text{if } n \equiv 1, 4 \pmod{9} \\ (-3, 3, 0, -2, 1, 1, 1, -2) \cdot \mathbf{v} & \text{if } n \equiv 2, 5 \pmod{9} \\ (3, 0, -3, 4, -2, 4, -2, -2) \cdot \mathbf{v} & \text{if } n \equiv 7 \pmod{9} \\ (-3, 3, 0, 4, -2, -2, -2, 4) \cdot \mathbf{v} & \text{if } n \equiv 7 \pmod{9} \end{cases} \\
F_3(n, 2, 1, 1) &= 3^{n-3} - \frac{1}{81} \cdot \begin{cases} (3, 0, -3, 4, -2, 4, -2, -2) \cdot \mathbf{v} & \text{if } n \equiv 1, 5 \pmod{9} \\ (3, 0, -3, -2, 1, -2, 1, 1) \cdot \mathbf{v} & \text{if } n \equiv 2, 4, 7, 8 \pmod{9} \end{cases} \\
F_3(n, 0, 2, 1) &= 3^{n-3} - \frac{1}{81} \cdot \begin{cases} (9, 6, 12, -2, 1, 1, -2, 1) \cdot \mathbf{v} & \text{if } n \equiv 1, 4, 7 \pmod{9} \\ (12, 9, 6, 4, 7, -5, -2, -2) \cdot \mathbf{v} & \text{if } n \equiv 2, 5, 8 \pmod{9} \end{cases}
\end{aligned}$$

$$\begin{aligned}
F_3(n, 1, 2, 1) &= 3^{n-3} - \frac{1}{81} \cdot \begin{cases} (0, -3, 3, 4, -2, -2, 4, -2) \cdot \mathbf{v} & \text{if } n \equiv 1, 8 \pmod{9} \\ (0, -3, 3, -2, 1, 1, -2, 1) \cdot \mathbf{v} & \text{if } n \equiv 2, 4, 5, 7 \pmod{9} \end{cases} \\
F_3(n, 2, 2, 1) &= 3^{n-3} - \frac{1}{81} \cdot \begin{cases} (0, -3, 3, -2, 1, 1, -2, 1) \cdot \mathbf{v} & \text{if } n \equiv 1, 7 \pmod{9} \\ (-3, 3, 0, -2, 1, 1, 1, -2) \cdot \mathbf{v} & \text{if } n \equiv 2, 8 \pmod{9} \\ (0, -3, 3, 4, -2, -2, 4, -2) \cdot \mathbf{v} & \text{if } n \equiv 4 \pmod{9} \\ (-3, 3, 0, 4, -2, -2, -2, 4) \cdot \mathbf{v} & \text{if } n \equiv 5 \pmod{9} \end{cases} \\
F_3(n, 0, 0, 2) &= 3^{n-3} - \frac{1}{81} (-21, -15, -18, 4, 7, 7, 7, 4) \cdot \mathbf{v} \quad \text{if } n \equiv 1, 2, 4, 5, 7, 8 \pmod{9} \\
F_3(n, 1, 0, 2) &= 3^{n-3} - \frac{1}{81} \cdot \begin{cases} (-3, 3, 0, -2, 1, 1, 1, -2) \cdot \mathbf{v} & \text{if } n \equiv 1, 4 \pmod{9} \\ (3, 0, -3, -2, 1, -2, 1, 1) \cdot \mathbf{v} & \text{if } n \equiv 2, 8 \pmod{9} \\ (3, 0, -3, 4, -2, 4, -2, -2) \cdot \mathbf{v} & \text{if } n \equiv 5 \pmod{9} \\ (-3, 3, 0, 4, -2, -2, -2, 4) \cdot \mathbf{v} & \text{if } n \equiv 7 \pmod{9} \end{cases} \\
F_3(n, 2, 0, 2) &= 3^{n-3} - \frac{1}{81} \cdot \begin{cases} (-3, 3, 0, -2, 1, 1, 1, -2) \cdot \mathbf{v} & \text{if } n \equiv 1, 7 \pmod{9} \\ (0, -3, 3, -2, 1, 1, -2, 1) \cdot \mathbf{v} & \text{if } n \equiv 2, 5 \pmod{9} \\ (-3, 3, 0, 4, -2, -2, -2, 4) \cdot \mathbf{v} & \text{if } n \equiv 4 \pmod{9} \\ (0, -3, 3, 4, -2, -2, 4, -2) \cdot \mathbf{v} & \text{if } n \equiv 8 \pmod{9} \end{cases} \\
F_3(n, 0, 1, 2) &= 3^{n-3} - \frac{1}{81} \cdot \begin{cases} (12, 9, 6, -2, 1, -2, 1, 1) \cdot \mathbf{v} & \text{if } n \equiv 1, 4, 7 \pmod{9} \\ (9, 6, 12, 4, 7, -2, -5, -2) \cdot \mathbf{v} & \text{if } n \equiv 2, 5, 8 \pmod{9} \end{cases} \\
F_3(n, 1, 1, 2) &= 3^{n-3} - \frac{1}{81} \cdot \begin{cases} (3, 0, -3, 4, -2, 4, -2, -2) \cdot \mathbf{v} & \text{if } n \equiv 1 \pmod{9} \\ (-3, 3, 0, -2, 1, 1, 1, -2) \cdot \mathbf{v} & \text{if } n \equiv 2, 8 \pmod{9} \\ (3, 0, -3, -2, 1, -2, 1, 1) \cdot \mathbf{v} & \text{if } n \equiv 4, 7 \pmod{9} \\ (-3, 3, 0, 4, -2, -2, -2, 4) \cdot \mathbf{v} & \text{if } n \equiv 5 \pmod{9} \end{cases} \\
F_3(n, 2, 1, 2) &= 3^{n-3} - \frac{1}{81} \cdot \begin{cases} (3, 0, -3, -2, 1, -2, 1, 1) \cdot \mathbf{v} & \text{if } n \equiv 1, 2, 4, 5 \pmod{9} \\ (3, 0, -3, 4, -2, 4, -2, -2) \cdot \mathbf{v} & \text{if } n \equiv 7, 8 \pmod{9} \end{cases} \\
F_3(n, 0, 2, 2) &= 3^{n-3} - \frac{1}{81} \cdot \begin{cases} (9, 6, 12, -2, 1, 1, -2, 1) \cdot \mathbf{v} & \text{if } n \equiv 1, 4, 7 \pmod{9} \\ (12, 9, 6, 4, 7, -5, -2, -2) \cdot \mathbf{v} & \text{if } n \equiv 2, 5, 8 \pmod{9} \end{cases} \\
F_3(n, 1, 2, 2) &= 3^{n-3} - \frac{1}{81} \cdot \begin{cases} (0, -3, 3, -2, 1, 1, -2, 1) \cdot \mathbf{v} & \text{if } n \equiv 1, 2, 7, 8 \pmod{9} \\ (0, -3, 3, 4, -2, -2, 4, -2) \cdot \mathbf{v} & \text{if } n \equiv 4, 5 \pmod{9} \end{cases} \\
F_3(n, 2, 2, 2) &= 3^{n-3} - \frac{1}{81} \cdot \begin{cases} (0, -3, 3, 4, -2, -2, 4, -2) \cdot \mathbf{v} & \text{if } n \equiv 1 \pmod{9} \\ (-3, 3, 0, -2, 1, 1, 1, -2) \cdot \mathbf{v} & \text{if } n \equiv 2, 5 \pmod{9} \\ (0, -3, 3, -2, 1, 1, -2, 1) \cdot \mathbf{v} & \text{if } n \equiv 4, 7 \pmod{9} \\ (-3, 3, 0, 4, -2, -2, -2, 4) \cdot \mathbf{v} & \text{if } n \equiv 8 \pmod{9} \end{cases}
\end{aligned}$$

6 Final Remarks and Open Problems

We now list a few open problems and research directions regarding the proposed algorithm and approach.

We first remark that although computing the set of varieties for a given $q = p^r$ and $\mathbf{f} = (T_{l_{m-1}}, \dots, T_{l_0})$ with $p > l_{m-1} > \dots > l_0 \geq 1$ is extremely simple, computing the characteristic values of these varieties is in general completely non-trivial. However, once they have been computed, the formulae are valid for all $n \geq l_{m-1}$ if $(n, p) = 1$, which of course is better than brute force testing the irreducibility of all 2^{n-m} such degree n polynomials, for n sufficiently large. The fact that such an algorithm – and hence formulae – exists for solving the prescribed coefficients problem exactly under the specified conditions is the central contribution of the present work.

There are many properties and consequences of the algorithm that have yet to be determined or explored. This work should therefore be considered as an introduction to the approach with explicit

proof-of-concept examples, which will hopefully lead to further developments and insights into the prescribed coefficients problem.

One obvious question is are the varieties obtained always curves? If so, are they always irreducible and how large are their genera? If they are all irreducible curves then one has the following application of Algorithm 1. For each $1 \leq i \leq q^m - 1$, suppose that the linearisation process of Transform 3 entails computing the L-polynomials of q^{s_i} curves $C_{i,j}$ for $1 \leq j \leq q^{s_i}$, each of genus $g_{i,j}$, so that $|\#C_{i,j}(\mathbb{F}_{q^n}) - (q^n + 1)| \leq 2g_{i,j}q^{n/2}$. Then since one must add the number of points of all q^{s_i} curves and divide by q^{s_i+1} to obtain $Z_1(\mathbf{i} \cdot \mathbf{f})$, one obtains the bound

$$|Z_1(\mathbf{i} \cdot \mathbf{f}) - (q^{n-1} + 1/q)| \leq 2 \left(\sum_{j=1}^{q^{s_i}} g_{i,j} \right) q^{n/2-s_i-1}.$$

Applying Transform 2 then gives

$$\#\{a \in \mathbb{F}_{q^n} \mid T_{l_{m-1}}(a) = t_{l_{m-1}}, \dots, T_{l_0}(a) = t_{l_0}\} = q^{n-m} + O(q^{n/2-m+1}),$$

with the q^{m-1} factor coming from the denominator of $S_{q,m}^{-1}$, while the implied constant depends on all of the $g_{i,j}$'s. Transform 1 then implies that the number of irreducibles with the same prescribed coefficients is $\frac{q^{n-m}}{n} + O(\frac{q^{n/2}}{n})$, as one would expect. Therefore, having bounds on the $g_{i,j}$ would lead to lower bounds needed on n for the existence of irreducibles with such prescribed coefficients.

Another natural question is whether it is possible to obviate the failure of Newton's identities in Transform 3 by transferring the problem to p -adic fields, à la Fan and Han's refinement [11] of Han's work on Cohen's problem [38]? This would allow one to circumvent the $l < p$ constraint and develop an analogue to Transform 3 for any number of prescribed coefficients in any position, from which Transforms 2 and 1 could be applied.

Finally, for binary fields, it would be informative to compute the L-polynomials for the $l = 6$ and $l = 7$ cases; although it is feasible to compute these by brute force point counting, it would be preferable to have a more elegant approach. Also, can one compute $F_2(n, t_1, t_2, t_3, t_4)$ for n even, and more generally give a method for resolving the $(n, p) = p$ cases for $q > 2$? Lastly, is it possible to express $T_l(\alpha + \beta)$ in terms of lower degree traces for any or all $l > 7$?

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